Magnetic Dipoles

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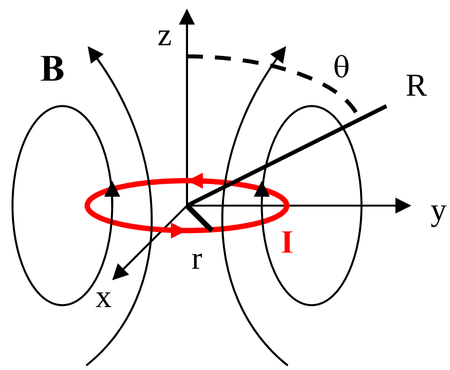
Magnetic Field of Current Loop



For distances $R \gg r$ (the loop radius), the calculation of the magnetic field does not depend on the shape of the current loop. It only depends on the current and the area (as well as R and θ):

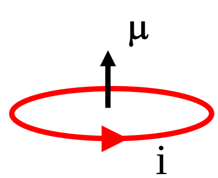
$$\mathbf{B} = \begin{cases} B_r = 2|\boldsymbol{\mu}| \dfrac{\mu_0}{4\pi} \dfrac{\cos\theta}{R^3} \\ B_\theta = |\boldsymbol{\mu}| \dfrac{\mu_0}{4\pi} \dfrac{\sin\theta}{R^3} \end{cases}$$

where $\boldsymbol{\mu} = i\mathbf{A}$ is the magnetic dipole moment of the loop

Here i is the current in the loop, A is the loop area, R is the radial distance from the center of the loop, and θ is the polar angle from the Z-axis. The field is equivalent to that from a tiny bar magnet (a magnetic dipole).

We define the **magnetic dipole moment** to be a vector pointing out of the plane of the current loop and with a magnitude equal to the product of the current and loop area:

$$\boxed{\vec{\mu} \equiv i\,\vec{A}}$$



The area vector, and thus the direction of the magnetic dipole moment, is given by a right-hand rule using the direction of the currents.

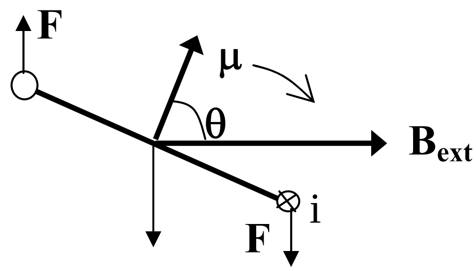
Interaction of Magnetic Dipoles in External Fields

Torque

By the $\mathbf{F} = i\mathbf{L} \times \mathbf{B}_{\text{ext}}$ force law, we know that a current loop (and thus a magnetic dipole) feels a torque when placed in an external magnetic field:

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}$$

The direction of the torque is to line up the dipole moment with the magnetic field:



Potential Energy

Since the magnetic dipole wants to line up with the magnetic field, it must have higher potential energy when it is aligned opposite to the magnetic field direction and lower potential energy when it is aligned with the field.

To see this, let us calculate the work done by the magnetic field when aligning the dipole. Let θ be the angle between the magnetic dipole direction and the external field direction.

$$\begin{aligned} W &= \int \mathbf{F} \cdot d\mathbf{s} \\ &= \int |\mathbf{F}| \sin\theta ds = -\int r|\mathbf{F}|\sin\theta \, d\theta \quad (\text{where } ds = -rd\theta) \\ &= -\int |\mathbf{r} \times \mathbf{F}| d\theta \\ \Rightarrow W &= -\int |\boldsymbol{\tau}| \, d\theta \end{aligned}$$

Now the potential energy of the dipole is the negative of the work done by the field:

$$U = -W = \int \tau d\theta$$

The zero-point of the potential energy is arbitrary, so let's take it to be zero when θ=90°. Then:

$$U = +\int_{\pi/2}^{\theta} \tau d\theta = +\int_{\pi/2}^{\theta} \mu B \sin\theta' d\theta'$$

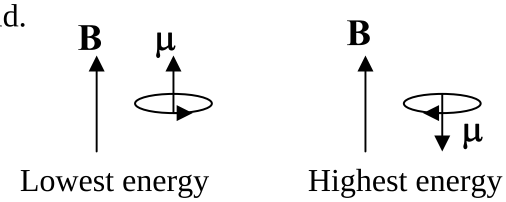
The positive sign arises because $\boldsymbol{\tau}\cdot d\boldsymbol{\theta} = -\tau d\theta$, τ and θ are oppositely aligned. Thus,

$$U = -\mu B\cos\theta\Big|_{\pi/2}^{\theta} = -\mu B\cos\theta$$

Or simply:

$$\boxed{U = -\boldsymbol{\mu}\cdot\mathbf{B}}$$

The lowest energy configuration is for $\boldsymbol{\mu}$ and $\mathbf{B}$ parallel. Work (energy) is required to re-align the magnetic dipole in an external B field.

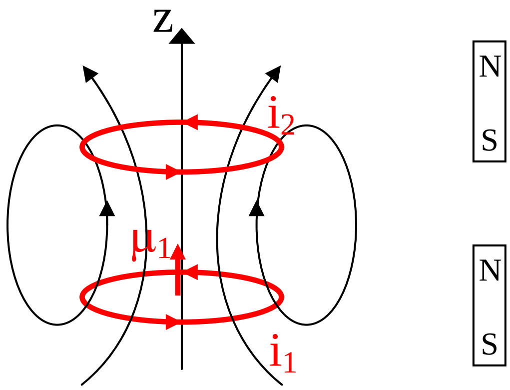


The change in energy required to flip a dipole from one alignment to the other is

$$\boxed{\Delta U = 2\mu B}$$

Force on a Magnetic Dipole in a Non-uniform Field (or why magnets stick!)

Two bar magnets stick together when opposite poles are brought together (north-south), and repel when the same poles are brought together (north-north, south-south). The magnetic field of a small bar magnet is equivalent to a small current loop, so two magnets stacked end-to-end vertically are equivalent to two current loops stacked:



The potential energy on one dipole from the magnetic field from the other is:

$$U = -\boldsymbol{\mu}_1 \cdot \mathbf{B}_2 = -\mu_{z1} B_{z2} \quad \text{(choosing the z-axis for the magnetic dipole moment)}$$

Now force is derived from the rate of change of the potential energy:

$$\mathbf{F} = -\nabla U = -\frac{\partial U}{\partial z}\hat{\mathbf{z}} \quad \text{(for this particular case)}$$

For example, the gravitational potential energy of a mass a distance z above the surface of the Earth can be expressed by $U = mgz$. Thus, the force is $\mathbf{F} = -mg\ \hat{\mathbf{z}}$ (i.e. down)

For the case of the stacked dipoles:

$$F_z = -\frac{\partial U}{\partial z} = \mu_{1z} \frac{\partial B_{2z}}{\partial z}$$

or in general, any magnetic dipole placed in a non-uniform B-field:

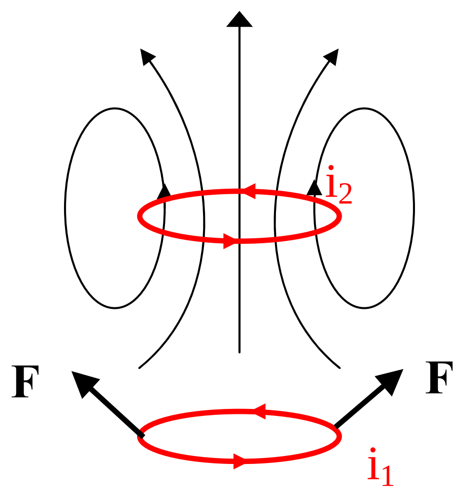
$$\boxed{F_z = \mu_z \frac{\partial B}{\partial z}}$$

Thus, there is a force acting on a dipole when placed in a non-uniform magnetic field.

For this example, the field from loop 2 increases with z as loop 1 is brought toward it from below: $\frac{\partial B_2}{\partial z} > 0$

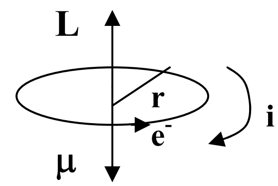
Thus, the force on loop 1 from the non-uniform field of loop 2 is directed up, and we see that there is an attractive force between them. North-South attract!

Another way to see this attraction is to consider the $\mathbf{F} = i\mathbf{L} \times \mathbf{B}_{\text{ext}}$ force acting on the current in loop 1 in the presence of the non-uniform field of loop 2:



Atomic Dipoles

Now why do some materials have magnetic fields in the absence of electric currents? Consider the atom. An electron "orbiting" an atom is like a small current loop. Let's consider first a classical model of the atom, with the electron in a circular orbit:



We can determine the magnetic dipole moment as follows:

$$\mu = i\,A = \frac{(-e)}{T}\pi\, r^2$$

$$T = \frac{2\pi\, r}{v} = \text{period}$$

$$\Rightarrow \mu = \frac{(-e)}{2\pi\, r} v\pi\, r^2 = \frac{-erv}{2}$$

But <u>angular momentum</u> is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

$$L = |\mathbf{L}| = rp = mvr \quad \text{for circular orbits}$$

This implies:

$$\boxed{\boldsymbol{\mu} = \frac{-e}{2m}\mathbf{L}}$$

The amazing thing is that this relation, which was defined classically, *also holds in quantum physics!* The details of the orbit are not important, only that there is some net angular momentum. An atom with an electron in an orbit with angular momentum is a small current loop, which implies that it is also a magnetic dipole.

Now consider an atom immersed in an external magnetic field applied along the *z*-axis. The potential energy is:

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} = -\mu_z B_z$$

and the magnetic dipole moment is:

$$\mu_z = \frac{-e}{2m} L_z$$

From quantum physics, we have that the average angular momentum about the z-axis is given by:

$$\langle L_z \rangle = \frac{m_\ell h}{2\pi}$$

where $m_\ell = 0, \pm 1, \pm 2, ...$ is an integer (the "quantum number"), and

$$h = 6.626 \times 10^{-34} \text{ J-s}$$

is a new constant in nature known as Planck's Constant.

Thus, for the potential energy of an atom in an external magnetic field, we have:

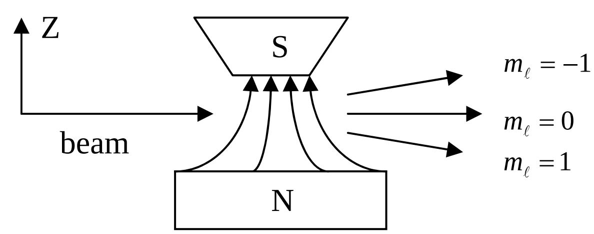
$$U = -\mu_B B \; m_\ell$$
$$\mu_B = \frac{eh}{4\pi m_e} = 5.79 \times 10^{-5} \text{ eV/T = Bohr magneton}$$

The Stern-Gerlach Experiment and the "Spin" of the Electron

Let's consider the Stern-Gerlach experiment of 1922. In that experiment, a *neutral* atomic beam is passed through an inhomogeneous magnetic field. (We use neutral atoms because otherwise there will be a Lorentz force on each atom: $\mathbf{F} = -q\,\mathbf{v} \times \mathbf{B}$). For such a field, there will be a force induced on the magnetic dipole moment of each atom despite the atom being neutral:

$$F_z = \mu_z \frac{\partial B}{\partial z}$$

where the field gradient is chosen to lie along the z-axis.



We thus expect to see the beam deflect depending on the value of μ_z. If the magnetic dipole moment is directed up ($\mu_z > 0$), then the force is up. If the magnetic dipole moment is directed down ($\mu_z < 0$), then the force is down.

Recall that:

$$\langle \mu_z \rangle = \frac{-e}{2m} \langle L_z \rangle = -\mu_B m_\ell$$

For the silver atoms used in the experiment, one would expect to see either no deflection, or three lines, or five, etc. depending on the value of ℓ for the orbital angular momentum of each atom. (Actually, silver has its outermost electron in an *s* state, so one would expect no deflection since ℓ=0 and thus $m_\ell = 0$ for the atom). One would definitely not expect to see only **two lines**, both of which are deflected from the straight-through direction! This is what was indeed seen, and something new was determined to be going on.

In 1925, Goudsmit and Uhlenbeck proposed that the electron itself must possess **intrinsic angular momentum**. That is, the electron is like a small bar magnet! As a bar magnet, it would possess a magnetic dipole moment, and would thus be deflected by the non-uniform magnetic field because of the force we derived earlier.

However, as far as we know from various measurements, the electron has no size at all. It is just a point. So we cannot think of an electron as a current loop arising from one charged object orbiting another, like we did for the atom.

Thus, this intrinsic angular momentum is called **spin**, and it is purely a quantum mechanical effect. The spin of the electron is apparently only ½, because only two states are observed (the quantum number m_s = ½, not an integer).

In the Stern-Gerlach experiment, each silver atom has $\ell = 0$, so the entire magnetic moment of the atom comes from the outer electron, which has *s* = ½. Thus, there are two possible values for the electron's magnetic dipole moment (neither of which are zero), and two beams are deflected through the inhomogeneous field. The experiment separates m_s = ½, from m_s = -½.

Now when an electron is placed in an external magnetic field, its potential energy based on the 2 possible orientations of the spin dipole moment is slightly changed from the form derived for the atomic dipole moment:

$$U = -2\mu_B B\, m_s \quad \left(m_s = \pm \frac{1}{2} \right)$$

Note that there is a factor 2 difference with respect to the earlier formula (the electron's "gyromagnetic ratio"), but that the value of m_s is a half and not an integer. So in effect the magnetic dipole moment of the electron is just μ_B.

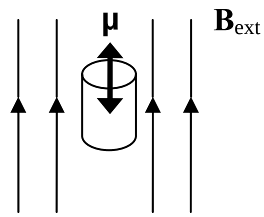
Magnetic Materials

So matter, a collection of atoms, is also a collection of atomic magnetic dipoles.

Magnetization ≡ net dipole moment per unit volume

$$\mathbf{M} = \frac{\vec{\mu}}{V} = \frac{\sum_{i=1}^{\infty} \vec{\mu}_i}{V}$$ where i is a sum over all atomic dipoles

Usually all dipoles in matter are aligned in random directions, so the net magnetization is zero: $\boldsymbol{M} = 0$.



However, if a material is placed into an external magnetic field, the dipoles will tend to align with or against the applied field, and the total magnetic field is then:

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{M}} = \mathbf{B}_{\text{ext}} + \mu_0 \mathbf{M}$$

In weak fields, $\mathbf{M} \propto \mathbf{B}_{\text{ext}}$, $\mu_0 \mathbf{M} = \chi \mathbf{B}_{\text{ext}}$

χ ≡ magnetic susceptibility of the material

Book defines magnetic permeability: $\kappa_M = 1 + \chi$

Materials are classified by their magnetic properties according to the following behavior of χ:

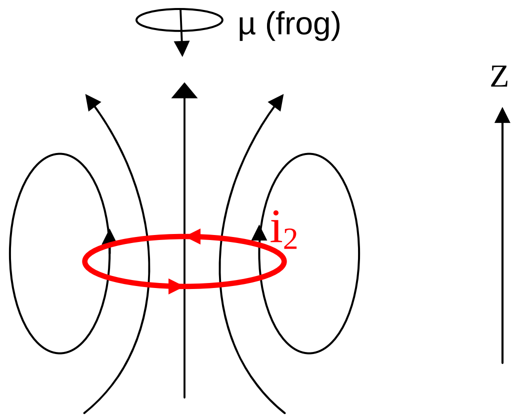
$\chi < 0$: diamagnetic
material reduces **B:** changes electron orbitals in materials without μ
e.g. water has $\chi \approx -5.7 \times 10^{-6}$

$\chi > 0$: paramagnetic
material enhances **B:** atomic dipoles align with applied field, $\chi \approx 10^{-7} - 10^{-2}$

$\chi >> 0$: ferromagnetic
material enhances **B**
enhances by factor of 1000 – 10000 !

Ferromagnetic substances will exhibit hysteresis, which means that as you raise and lower the strength of the external magnetic field, the magnetization will not follow the same curve in both directions. For example, when the external field returns to zero, the net magnetization may not be zero (and the sign will depend on whether you were raising or lowering the field).

Frog in a magnetic field:



A frog, as all living animals, is mostly made of water so it is diamagnetic. Thus its induced dipole moment aligns opposite that of an applied field. Suppose a frog sits above the fringe field of a solenoid magnet. Let's solve for the force on the frog using what we learned earlier:

$$F_z = \mu_z \frac{\partial B}{\partial z}$$

$$\mu_z < 0 \;\; \text{(anti-aligned with field)}$$

$$\frac{\partial B}{\partial z} < 0 \;\; \text{(decreasing vertically)}$$

$$\Rightarrow F_z > 0 \;\; \text{(vertical)}$$

For a strong enough gradient in the solenoid magnetic field, the force is nough to overcome that of gravity. The frog will levitate in the field! (and if the magnet were wide enough, so would you.)