Mathematical analysis of random noise

by

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Introduction

THIS paper deals with the mathematical analysis of noise obtained by passing random noise through physical devices. The random noise considered is that which arises from shot effect in vacuum tubes or from thermal agitation of electrons in resistors. Our main interest is in the statistical properties of such noise and we leave to one side many physical results of which Nyquist's law may be given as an example.[1]

About half of the work given here is believed to be new, the bulk of the new results appearing in Parts III and IV. In order to provide a suitable introduction to these results and also to bring out their relation to the work of others, this paper is written as an exposition of the subject indicated in the title.

When a broad band of random noise is applied to some physical device, such as an electrical network, the statistical properties of the output are often of interest. For example, when the noise is due to shot effect, its mean and standard deviations are given by Campbell's theorem (Part I) when the physical device is linear. Additional information of this sort is given by the (auto) correlation function which is a rough measure of the dependence of values of the output separated by a fixed time interval.

The paper consists of four main parts. The first part is concerned with shot effect. The shot effect is important not only in its own right but also because it is a typical source of noise. The Fourier series representation of a noise current, which is used extensively in the following parts, may be obtained from the relatively simple concepts inherent in the shot effect.

The second part is devoted principally to the fundamental result that the power spectrum of a noise current is the Fourier transform of its correlation function. This result is used again and again in Parts III and IV.

A rather thorough discussion of the statistics of random noise currents is given in Part III. Probability distributions associated with the maxima of the current and the maxima of its envelope are developed. Formulas for the expected number of zeros and maxima per second are given, and a start is made towards obtaining the probability distribution of the zeros.

When a noise voltage or a noise voltage plus a signal is applied to a non-

[1] An account of this field is given by E. B. Moullin, "Spontaneous Fluctuations of Voltage," Oxford (1938).

linear device, such as a square-law or linear rectifier, the output will also contain noise. The methods which are available for computing the amount of noise and its spectral distribution are discussed in Part IV.

Acknowledgement

I wish to thank my friends for many helpful suggestions and discussions regarding the subject of this paper. Although it has been convenient to acknowledge some of this assistance in the text, I appreciate no less sincerely the considerable amount which is not mentioned. In particular, I am indebted to Miss Darville for computing the curves in Parts III and IV.

Summary of Results

Before proceeding to the main body of the paper, we shall state some of the principal results. It is hoped that this summary will give the casual reader an over-all view of the material covered and at the same time guide the reader who is interested in obtaining some particular item of information to those portions of the paper which may possibly contain it.

Part I—Shot Effect

Shot effect noise results from the superposition of a great number of disturbances which occur at random. A large class of noise generators produce noise in this way.

Suppose that the arrival of an electron at the anode of the vacuum tube at time $t = 0$ produces an effect $F(t)$ at some point in the output circuit. If the output circuit is such that the effects of the various electrons add linearly, the total effect at time t due to all the electrons is

$$I(t) = \sum_{k=-\infty}^{+\infty} F(t - t_k) \qquad (1.2\text{–}1)$$

where the k^{th} electron arrives at t_k and the series is assumed to converge. Although the terminology suggests that $I(t)$ is a current, and it will be spoken of as a noise current, it may be any quantity expressible in the form (1.2–1).

1. Campbell's theorem: The average value of $I(t)$ is

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \qquad (1.2\text{–}2)$$

and the mean square value of the fluctuation about this average is

$$\text{ave. } [I(t) - \overline{I(t)}]^2 = \nu \int_{-\infty}^{+\infty} F^2(t)\, dt \qquad (1.2\text{–}3)$$

where ν is the average number of electrons arriving per second at the anode. In this expression the electrons are supposed to arrive independently and at random. $\nu e^{-\nu t}\, dt$ is the probability that the length of the interval between two successive arrivals lies between t and $t + dt$.

2. Generalization of Campbell's theorem. Campbell's theorem gives information about the average value and the standard deviation of the probability distribution of $I(t)$. A generalization of the theorem gives information about the third and higher order moments. Let

$$I(t) = \sum_{-\infty}^{+\infty} a_k F(t - t_k) \qquad (1.5\text{-}1)$$

where $F(t)$ and t_k are of the same nature as those in (1.2-1) and $\cdots a_1$, $a_2, \cdots a_k, \cdots$ are independent random variables all having the same distribution. Then the n^{th} semi-invariant of the probability density $P(I)$ of $I = I(t)$ is

$$\lambda_n = \nu \overline{a^n} \int_{-\infty}^{+\infty} [F(t)]^n \, dt \qquad (1.5\text{-}2)$$

The semi-invariants are defined as the coefficients in the expansion of the characteristic function $f(u)$:

$$\log_e f(u) = \sum_{n=1} \frac{\lambda_n}{n!} (iu)^n \qquad (1.5\text{-}3)$$

where

$$f(u) = \text{ave. } e^{iIu} = \int_{-\infty}^{+\infty} P(I) e^{iIu} \, dI$$

The moments may be computed from the λ's.

3. As $\nu \to \infty$ the probability density $P(I)$ of the shot effect current approaches a normal law. The way it is approached is given by

$$P(I) \sim \sigma^{-1} \varphi^{(0)}(x) - \frac{\lambda_3 \sigma^{-4}}{3!} \varphi^{(3)}(x) + \left[\frac{\lambda_4 \sigma^{-5}}{4!} \varphi^{(4)}(x) + \frac{\lambda_3^2 \sigma^{-7}}{72} \varphi^{(6)}(x) \right] + \cdots \qquad (1.6\text{-}3)$$

where the λ's are given by (1.5-2) and

$$\sigma^2 = \lambda_2 \qquad x = \frac{I - \bar{I}}{\sigma} \qquad \varphi^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dx^n} e^{-x^2/2}$$

Since the λ's are of the order of ν, σ is of the order of $\nu^{1/2}$ and the orders of σ^{-1}, $\lambda_3\sigma^{-4}$, $\lambda_4\sigma^{-5}$ and $\lambda_3^2\sigma^{-7}$ are $\nu^{-1/2}$, ν^{-1}, $\nu^{-3/2}$ and $\nu^{-3/2}$ respectively. A possible use of this result is to determine whether a noise due to random independent events occuring at the rate of ν per second may be regarded as "random noise" in the sense of this work.

4. When $I(t)$, as given by (1.5-1), is analyzed as a Fourier series over an interval of length T a set of Fourier coefficients is obtained. By taking many different intervals, all of length T, many sets of coefficients are obtained. If ν is sufficiently large these coefficients tend to be distributed normally and independently. A discussion of this is given in section 1.7.

Part II—Power Spectra and Correlation Functions

1. Suppose we have a curve, such as an oscillogram of a noise current, which extends from $t = 0$ to $t = \infty$. Let this curve be denoted by $I(t)$. The correlation function of $I(t)$ is $\psi(\tau)$ which is defined as

$$\psi(\tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \tag{2.1-4}$$

where the limit is assumed to exist. This function is closely connected with another function, the power spectrum, $w(f)$, of $I(t)$. $I(t)$ may be regarded as composed of many sinusoidal components. If $I(t)$ were a noise current and if it were to flow through a resistance of one ohm the average power dissipated by those components whose frequencies lie between f and $f + df$ would be $w(f)\, df$.

The relation between $w(f)$ and $\psi(\tau)$ is

$$w(f) = 4 \int_0^\infty \psi(\tau) \cos 2\pi f\tau\, d\tau \tag{2.1-5}$$

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1-6}$$

When $I(t)$ has no d.c. or periodic components,

$$w(f) = \underset{T\to\infty}{\text{Limit}} \frac{2\,|\,S(f)\,|^2}{T} \tag{2.1-3}$$

where

$$S(f) = \int_0^T I(t)e^{-2\pi ift}\, dt.$$

The correlation function for

$$I(t) = A + C \cos (2\pi f_0 t - \varphi)$$

is

$$\psi(\tau) = A^2 + \frac{C^2}{2} \cos 2\pi f_0 \tau \tag{2.2-3}$$

These results are discussed in sections 2.1 to 2.4 inclusive.

2. So far we have supposed $I(t)$ to be some definite function for which a curve may be drawn. Now consider $I(t)$ to be given by a mathematical expression into which, besides t, a number of parameters enter. $w(f)$ and $\psi(\tau)$ are now obtained by averaging the integrals over the possible values of the parameters. This is discussed in section 2.5.

3. The correlation function for the shot effect current of (1.2–1) is

$$\psi(\tau) = \nu \int_{-\infty}^{+\infty} F(t)F(t+\tau)\,dt + \left[\nu \int_{-\infty}^{+\infty} F(t)\,dt\right]^2 \qquad (2.6\text{–}2)$$

The distributed portion of the power spectrum is

$$w_1(f) = 2\nu \mid s(f) \mid^2$$

where

$$s(f) = \int_{-\infty}^{+\infty} F(t)e^{-2\pi i f t}\,dt \qquad (2.6\text{–}5)$$

The complete power spectrum has in addition to $w_1(f)$ an impulse function representing the d.c. component $\overline{I(t)}$.

In the formulas above for the shot effect it was assumed that the expected number, ν, of electrons per second did not vary with time. A case in which ν does vary with time is briefly discussed near the end of Section 2.6.

4. Random telegraph signal. Let $I(t)$ be equal to either a or $-a$ so that it is of the form of a flat top wave, and let the lengths of the tops and bottoms be distributed independently and exponentially. The correlation function and power spectrum of I are

$$\psi(\tau) = a^2 e^{-2\mu|\tau|} \qquad (2.7\text{–}4)$$

$$w(f) = \frac{2a^2\mu}{\pi^2 f^2 + \mu^2} \qquad (2.7\text{–}5)$$

where μ is the expected number of changes of sign per second.

Another type of random telegraph signal may be formed as follows: Divide the time scale into intervals of equal length h. In an interval selected at random the value of $I(t)$ is independent of the value in the other intervals and is equally likely to be $+a$ or $-a$. The correlation function of $I(t)$ is zero for $|\tau| > h$ and is

$$a^2\left(1 - \frac{|\tau|}{h}\right)$$

for $0 \leq |\tau| < h$ and the power spectrum is

$$w(f) = 2h\left(\frac{a \sin \pi f h}{\pi f h}\right)^2 \qquad (2.7\text{–}9)$$

5. There are two representations of a random noise current which are especially useful. The first one is

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \tag{2.8-1}$$

where a_n and b_n are independent random variables which are distributed normally about zero with the standard deviation $\sqrt{w(f_n)\Delta f}$ and where

$$\omega_n = 2\pi f_n, \qquad f_n = n\Delta f$$

The second one is

$$I(t) = \sum_{n=1}^{N} c_n \cos (\omega_n t - \varphi_n) \tag{2.8-6}$$

where φ_n is a random phase angle distributed uniformly over the range $(0, 2\pi)$ and

$$c_n = [2w(f_n)\Delta f]^{1/2}$$

At an appropriate point in the analysis N and Δf are made to approach infinity and zero, respectively, in such a manner that the entire frequency band is covered by the summations (which then become integrations).

6. The normal distribution in several variables and the central limit theorem are discussed in sections 2.9 and 2.10.

Part III—Statistical Properties of Noise Current

1. The noise current is distributed normally. This has already been discussed in section 1.6 for the shot-effect. It is discussed again in section 3.1 using the concepts introduced in Part II, and the assumption, used throughout Part III, that the average value of the noise current $I(t)$ is zero. The probability that $I(t)$ lies between I and $I + dI$ is

$$\frac{dI}{\sqrt{2\pi\psi_0}} e^{-I^2/2\psi_0} \tag{3.1-3}$$

where ψ_0 is the value of the correlation function, $\psi(\tau)$, of $I(t)$ at $\tau = 0$

$$\psi_0 = \psi(0) = \int_0^\infty w(f)\, df, \tag{3.1-2}$$

$w(f)$ being the power spectrum of $I(t)$. ψ_0 is the mean square value of $I(t)$, i.e., the r.m.s. value of $I(t)$ is $\psi_0^{1/2}$.

The characteristic function (ch. f.) of this distribution is

$$\text{ave. } e^{iuI(t)} = \exp - \frac{\psi_0}{2} u^2 \tag{3.1-6}$$

2. The probability that $I(t)$ lies between I_1 and $I_1 + dI$, and $I(t + \tau)$ lies between I_2 and $I_2 + dI_2$ when t is chosen at random is

$$[\psi_0^2 - \psi_\tau^2]^{-1/2} \frac{dI_1\, dI_2}{2\pi} \exp\left[\frac{-\psi_0 I_1^2 - \psi_0 I_2^2 + 2\psi_\tau I_1 I_2}{2(\psi_0^2 - \psi_\tau^2)}\right] \qquad (3.2\text{-}4)$$

where ψ_τ is the correlation function $\psi(\tau)$ of $I(t)$:

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \, df \qquad (3.2\text{-}3)$$

The ch. f. for this distribution is

$$\text{ave. } e^{iuI(t)+ivI(t+\tau)} = \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \qquad (3.2\text{-}7)$$

3. The expected number of zeros per second of $I(t)$ is

$$\frac{1}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} = 2\left[\frac{\int_0^\infty f^2 w(f)\, df}{\int_0^\infty w(f)\, df}\right]^{1/2} \qquad (3.3\text{-}11)$$

assuming convergence of the integrals. The primes denote differentiation with respect to τ:

$$\psi''(\tau) = \frac{d^2}{d\tau^2}\psi(\tau).$$

For an ideal band-pass filter whose pass band extends from f_a to f_b the expected number of zeros per second is

$$2\left[\frac{1}{3}\frac{f_b^3 - f_a^3}{f_b - f_a}\right]^{1/2} \qquad (3.3\text{-}12)$$

When f_a is zero this becomes $1.155\, f_b$ and when f_a is very nearly equal to f_b it approaches $f_b + f_a$.

4. The problem of determining the distribution function for the length of the interval between two successive zeros of $I(t)$ seems to be quite difficult. In section 3.4 some related results are given which lead, in some circumstances, to approximations to the distribution. For example, for an ideal narrow band-pass filter the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{d\tau}{2} \frac{a}{[1 + a^2(\tau - \tau_1)^2]^{3/2}}$$

where

$$a = \sqrt{3}\,\frac{(f_b + f_a)^2}{f_b - f_a}, \qquad \tau_1 = \frac{1}{f_b + f_a}$$

f_b and f_a being the upper and lower cut-off frequencies.

5. In section 3.5 several multiple integrals which occur in the work of Part III are discussed.

6. The distribution of the maxima of $I(t)$ is discussed in section 3.6. The expected number of maxima per second is

$$\frac{1}{2\pi}\left[-\frac{\psi_0^{(4)}}{\psi_0''}\right]^{1/2} = \left[\frac{\int_0^\infty f^4 w(f)\,df}{\int_0^\infty f^2 w(f)\,df}\right]^{1/2} \tag{3.6-6}$$

For a band-pass filter the expected number of maxima per second is

$$\left[\frac{3}{5}\,\frac{f_b^5 - f_a^5}{f_b^3 - f_a^3}\right]^{1/2} \tag{3.6-7}$$

For a low-pass filter where $f_a = 0$ this number is $0.775\,f_b$.

The expected number of maxima per second lying above the line $I(t) = I_1$ is approximately, when I_1 is large,

$$e^{-I_1^2/2\psi_0} \times \tfrac{1}{2}[\text{the expected number of zeros of } I \text{ per second}] \tag{3.6-11}$$

where ψ_0 is the mean square value of $I(t)$.

For a low-pass filter the probability that a maximum chosen at random from the universe of maxima lies between I and $I + dI$ is approximately, when I is large,

$$\frac{\sqrt{5}}{3}\,y e^{-y^2/2}\,\frac{dI}{\psi_0^{1/2}} \tag{3.6-9}$$

where

$$y = \frac{I}{\psi_0^{1/2}}$$

7. When we pass noise through a relatively narrow band-pass filter one of the most noticeable features of an oscillogram of the output current is its fluctuating envelope. In sections 3.7 and 3.8 some statistical properties of this envelope, denoted by R or $R(t)$, are derived.

The probability that the envelope lies between R and $R + dR$ is

$$\frac{R}{\psi_0}\,e^{-R^2/2\psi_0}\,dR \tag{3.7-10}$$

where ψ_0 is the mean square value of $I(t)$. The probability that $R(t)$ lies between R_1 and $R_1 + dR_1$ and at the same time $R(t + \tau)$ lies between R_2 and $R_2 + dR_2$ when t is chosen at random is obtained by multiplying (3.7-13) by $dR_1\, dR_2$. For an ideal band-pass filter, the expected number of maxima of the envelope in one second is

$$.64110(f_b - f_a) \tag{3.8-15}$$

When R is large, say $y > 2.5$ where

$$y = \frac{R}{\psi_0^{1/2}}, \qquad \psi_0^{1/2} = \text{r.m.s. value of } I(t),$$

the probability that a maximum of the envelope, selected at random from the universe of such maxima, lies between R and $R + dR$ is approximately

$$1.13(y^2 - 1)e^{-y^2/2}\frac{dR}{\psi_0^{1/2}}$$

A curve for the corresponding probability density is shown for the range $0 \leq y \leq 4$. Curves which compare the distribution function of the maxima of R with other distribution functions of the same type are also given.

8. In section 3.9 some information is given regarding the statistical behavior of the random variable:

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt \tag{3.9-1}$$

where t_1 is chosen at random and $I(t)$ is a noise current with the power spectrum $w(f)$ and the correlation function $\psi(\tau)$. The average value m_T of E is $T\psi_0$ and its standard deviation σ_T is given by (3.9-9). For a relatively narrow band-pass filter

$$\frac{\sigma_T}{m_T} \sim \frac{1}{\sqrt{T(f_b - f_a)}}$$

when $T(f_b - f_a) \gg 1$. This follows from equation (3.9-10). An expression which is believed to approximate the distribution of E is given by (3.9-20).

9. In section 3.10 the distribution of a noise current plus one or more sinusoidal currents is discussed. For example, if I consists of two sine waves plus noise:

$$I = P\cos pt + Q\cos qt + I_N, \tag{3.10-20}$$

where p and q are incommensurable and the r.m.s. value of the noise current I_N is $\psi_0^{1/2}$, the probability density of the envelope R is

$$R\int_0^\infty rJ_0(Rr)J_0(Pr)J_0(Qr)e^{-\psi_0 r^2/2}\, dr \tag{3.10-21}$$

where $J_0(\ )$ is a Bessel function.

Curves showing the probability density and distribution function of R, when $Q = 0$, for various ratios of P/r.m.s. I_N are given.

10. In section 3.11 it is pointed out that the representations (2.8-1) and (2.8-6) of the noise current as the sum of a great number of sinusoidal components are not the only ones which may be used in deriving the results given in the preceding sections of Part III. The shot effect representation

$$I(t) = \sum_{-\infty}^{+\infty} F(t - t_k)$$

studied in Part I may also be used.

Part IV—Noise Through Non-Linear Devices

1. Suppose that the power spectrum of the voltage V applied to the square-law device

$$I = \alpha V^2 \tag{4.1-1}$$

is confined to a relatively narrow band. The total low-frequency output current $I_{t\ell}$ may be expressed as the sum

$$I_{t\ell} = I_{dc} + I_{\ell f} \tag{4.1-2}$$

where I_{dc} is the d.c. component and $I_{\ell f}$ is the variable component. When none of the low-frequency band is eliminated (by audio frequency filters)

$$I_{t\ell} = \frac{\alpha R^2}{2} \tag{4.1-6}$$

where R is the envelope of V. If V is of the form

$$V = V_N + P \cos pt + Q \cos qt, \tag{4.1-4}$$

where V_N is a noise voltage whose mean square value is ψ_0, then

$$I_{dc} = \alpha\left(\psi_0 + \frac{P^2}{2} + \frac{Q^2}{2}\right)$$

$$\overline{I_{\ell f}^2} = \alpha^2\left[\psi_0^2 + P^2\psi_0 + Q^2\psi_0 + \frac{P^2Q^2}{2}\right] \tag{4.1-16}$$

2. If instead of a square-law device we have a linear rectifier,

$$I = \begin{cases} 0 & V < 0 \\ \alpha V, & V > 0 \end{cases} \tag{4.2-1}$$

the total low-frequency output is

$$I_{t\ell} = \frac{\alpha R}{\pi} \tag{4.2-2}$$

When V is a sine wave plus noise, $V_N + P\cos pt$,

$$I_{dc} = \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2} {}_1F_1(-\tfrac{1}{2};\, 1;\, -x) \qquad (4.2\text{-}3)$$

$$\overline{I_{lf}^2} = \frac{\alpha^2}{\pi^2}(P^2 + 2\psi_0) \qquad (4.2\text{-}6)$$

where ${}_1F_1$ is a hypergeometric function and

$$x = \frac{P^2}{2\psi_0} = \frac{\text{Ave. sine wave power}}{\text{Ave. noise power}} \qquad (4.2\text{-}4)$$

When x is large

$$\overline{I_{lf}^2} \sim \frac{\alpha^2\psi_0}{\pi^2}\left[1 - \frac{1}{4x}\cdots\right] \qquad (4.2\text{-}7)$$

If V consists of two sine waves plus noise, I_{dc} consists of a hypergeometric function of two variables. The equations running from (4.2-9) to (4.2-15) are concerned with this case. About the only simple equation is

$$\overline{I_{lf}^2} = \frac{\alpha^2}{\pi^2}[2\psi_0 + P^2 + Q^2] \qquad (4.2\text{-}14)$$

3. The expressions (4.1-6) and (4.2-2) for I_{lf} in terms of the envelope R of V, namely

$$\frac{\alpha R^2}{2} \quad \text{and} \quad \frac{\alpha R}{\pi},$$

are special cases of a more general result

$$I_{lf} = A_0(R) = \frac{1}{2\pi}\int_C F(iu)J_0(uR)\,du. \qquad (4.3\text{-}11)$$

In this expression $J_0(uR)$ is a Bessel function. The path of integration C and the function $F(iu)$ are chosen so that the relation between I and V may be expressed as

$$I = \frac{1}{2\pi}\int_C F(iu)e^{iVu}\,du. \qquad (4A\text{-}1)$$

A table giving $F(iu)$ and C for a number of common non-linear devices is shown in Appendix 4A.

If this relation is used to study the biased linear rectifier.

$$I = \begin{cases} 0, & V < B \\ V - B, & V > B \end{cases}$$

for the case in which V is $V_N + P \cos pt$, we find

$$I_{dc} \sim -\frac{B}{2} + \frac{P}{\pi} + \frac{B^2 + \psi_0}{2\pi P}$$

$$\overline{I_{lf}^2} \sim \frac{P^2 - B^2}{\pi^2 P^2} \psi_0 \tag{4.3-17}$$

when $P \gg |B|$, $P^2 \gg \psi_0$ where ψ_0 is the mean square value of V_N.

4. When V is confined to a relatively narrow band and there are no audio-frequency filters, the probability density and all the associated statistical properties of I_{lf} may be obtained by expressing I_{lf} as a function of the envelope R of V and then using the probability density of R. When V is $V_N + P \cos pt + Q \cos qt$ this probability density is given by the integral, (3.10-21) (which is the integral containing three Bessel functions stated in the above summary of Part III). When V consists of three sine waves plus noise there are four J_0's in the integrand, and so on. Expressions for $\overline{R^n}$ when R has the above distribution are given by equations (3.10-25) and (3.10-27).

When audio-frequency filters remove part of the low-frequency band the statistical properties, except the mean square value, of the resulting current are hard to compute. In section 4.3 it is shown that as the output band is chosen narrower and narrower, the statistical properties of the output current approach those of a random noise current.

5. The sections in Part IV from 4.4 onward are concerned with the problem: Given a non-linear device and an input voltage consisting of noise alone or of a signal plus noise. What is the power spectrum of the output? A survey of the methods available for the solution of this problem is given in section 4.4.

6. When a noise voltage V_N with the power spectrum $w(f)$ is applied to the square-law device

$$I = \alpha V^2 \tag{4.1-1}$$

the power spectrum of the output current I is, when $f \neq 0$,

$$W(f) = \alpha^2 \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \tag{4.5-5}$$

where $w(-x)$ is defined to equal $w(x)$. The power spectrum of I when V is either $P \cos pt + V_N$ or

$$Q(1 + k \cos pt) \cos qt + V_N$$

is considered in the portion of section 4.5 containing equations (4.5-10) to (4.5-17).

7. A method discovered independently by Van Vleck and North shows that the correlation function $\Psi(\tau)$ of the output current for an unbiased linear rectifier is

$$\Psi(\tau) = \frac{\psi_\tau}{4} + \frac{\psi_0}{2} {}_2F_1\left[-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \frac{\psi_\tau^2}{\psi_0^2}\right] \tag{4.7-6}$$

where the input voltage is V_N. The correlation function $\psi(\tau)$ of V_N is denoted by ψ_τ and the mean square value of V_N is ψ_0. The power spectrum $W(f)$ of I may be obtained from

$$W(f) = 4 \int_0^\infty \Psi(\tau) \cos 2\pi f\tau \, d\tau \tag{4.6-1}$$

by expanding the hypergeometric function and integrating termwise using

$$G_n(f) = \int_0^\infty \psi_\tau^n \cos 2\pi f\tau \, d\tau. \tag{4C-1}$$

Appendix 4C is devoted to the problem of evaluating the integral for $G_n(f)$.

8. Another method of obtaining the correlation function $\psi(\tau)$ of I, termed the "characteristic function method," is explained in section 4.8. It is illustrated in section 4.9 where formulas for $\Psi(\tau)$ and $W(f)$ are developed when the voltage $P \cos pt + V_N$ is applied to a general non-linear device.

9. Several miscellaneous results are given in section 4.10. The characteristic function method is used to obtain the correlation function for a square-law device. The general formulas of section 4.9 are applied to the case of a ν^{th} law rectifier when the input noise spectrum has a normal law distribution. Some remarks are also made concerning the audio-frequency output of a linear rectifier when the input voltage V is

$$Q(1 + r \cos pt) \cos qt + V_N.$$

10. A discussion of the hypergeometric function ${}_1F_1(a; c; x)$, which often occurs in problems concerning a sine wave plus noise, is given in Appendix 4B.

PART I

THE SHOT EFFECT

The shot effect in vacuum tubes is a typical example of noise. It is due to fluctuations in the intensity of the stream of electrons flowing from the cathode to the anode. Here we analyze a simplified form of the shot effect.

1.1 The Probability of Exactly K Electrons Arriving at the Anode in Time T

The fluctuations in the electron stream are supposed to be random. We shall treat this randomness as follows. We count the number of electrons flowing in a long interval of time T measured in seconds. Suppose there are K_1. Repeating this counting process for many intervals all of length T gives a set of numbers $K_2, K_3 \cdots K_M$ where M is the total number of intervals. The average number ν, of electrons per second is defined as

$$\nu = \lim_{M \to \infty} \frac{K_1 + K_2 + \cdots + K_M}{MT} \qquad (1.1\text{-}1)$$

where we assume that this limit exists. As M is increased with T being held fixed some of the K's will have the same value. In fact, as M increases the number of K's having any particular value will tend to increase. This of course is based on the assumption that the electron stream is a steady flow upon which random fluctuations are superposed. The probability of getting K electrons in a given trial is defined as

$$p(K) = \lim_{M \to \infty} \frac{\text{Number of trials giving exactly } K \text{ electrons}}{M} \qquad (1.1\text{-}2)$$

Of course $p(K)$ also depends upon T. We assume that the randomness of the electron stream is such than the probability that an electron will arrive at the anode in the interval $(t, t + \Delta t)$ is $\nu\Delta t$ where Δt is such that $\nu\Delta t \ll 1$, and that this probability is independent of what has happened before time t or will happen after time $t + \Delta t$.

This assumption is sufficient to determine the expression for $p(K)$ which is

$$p(K) = \frac{(\nu T)^K}{K!} e^{-\nu T} \qquad (1.1\text{-}3)$$

This is the "law of small probabilities" given by Poisson. One method of derivation sometimes used can be readily illustrated for the case $K = 0$. Thus, divide the interval, $(0, T)$ into M intervals each of length $\Delta t = \frac{T}{M}$. Δt is taken so small that $\nu\Delta t$ is much less than unity. (This is the "small probability" that an electron will arrive in the interval Δt). The probability that an electron will not arrive in the first sub-interval is $(1 - \nu\Delta t)$. The probability that one will not arrive in either the first or the second sub-interval is $(1 - \nu\Delta t)^2$. The probability that an electron will not arrive in any of the M intervals is $(1 - \nu\Delta t)^M$. Replacing M by $T/\Delta t$ and letting $\Delta t \to 0$ gives

$$p(0) = e^{-\nu T}$$

The expressions for $p(1), p(2), \cdots p(K)$ may be derived in a somewhat similar fashion.

1.2 STATEMENT OF CAMPBELL'S THEOREM

Suppose that the arrival of an electron at the anode at time $t = 0$ produces an effect $F(t)$ at some point in the output circuit. If the output circuit is such that the effects of the various electrons add linearly, the total effect at time t due to all the electrons is

$$I(t) = \sum_{k=-\infty}^{+\infty} F(t - t_k) \tag{1.2-1}$$

where the k^{th} electron arrives at t_k and the series is assumed to converge.

Campbell's theorem[2] states that the average value of $I(t)$ is

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.2-2}$$

and the mean square value of the fluctuation about this average is

$$\overline{(I(t) - \overline{I(t)})^2} = \nu \int_{-\infty}^{+\infty} F^2(t)\, dt \tag{1.2-3}$$

where ν is the average number of electrons arriving per second.

The statement of the theorem is not precise until we define what we mean by "average". From the form of the equations the reader might be tempted to think of a time average; e.g. the value

$$\operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T I(t)\, dt \tag{1.2-4}$$

However, in the proof of the theorem the average is generally taken over a great many intervals of length T with t held constant. The process is somewhat similar to that employed in (1.1) and in order to make it clear we take the case of $\overline{I(t)}$ for illustration. We observe $I(t)$ for many, say M, intervals each of length T where T is large in comparison with the interval over which the effect $F(t)$ of the arrival of a single electron is appreciable. Let ${}_nI(t')$ be the value of $I(t)$, t' seconds after the beginning of the n^{th} interval. t' is equal to t plus a constant depending upon the beginning time of the interval. We put the subscript in front because we wish to reserve the usual place for another subscript later on. The value of $\overline{I(t')}$ is then defined as

$$\overline{I(t')} = \operatorname*{Limit}_{M \to \infty} \frac{1}{M} [{}_1I(t') + {}_2I(t') + \cdots + {}_MI(t')] \tag{1.2-5}$$

and this limit is assumed to exist. The mean square value of the fluctuation of $I(t')$ is defined in much the same way.

[2] *Proc. Camb. Phil. Soc.* 15 (1909), 117–136, 310–328. Our proof is similar to one given by J. M. Whittaker, *Proc. Camb. Phil. Soc.* 33 (1937), 451–458.

Actually, as the equations (1.2-2) and (1.2-3) of Campbell's theorem show, these averages and all the similar averages encountered later turn out to be independent of the time. When this is true and when the M intervals in (1.2-5) are taken consecutively the time average (1.2-4) and the average (1.2-5) become the same. To show this we multiply both sides of (1.2-5) by dt' and integrate from 0 to T:

$$
\begin{aligned}
I(t') &= \underset{M\to\infty}{\text{Limit}} \frac{1}{MT} \sum_{m=1}^{M} \int_0^T {}_mI(t')\, dt' \\
&= \underset{M\to\infty}{\text{Limit}} \frac{1}{MT} \int_0^{MT} I(t)\, dt
\end{aligned}
\qquad (1.2\text{-}6)
$$

and this is the same as the time average (1.2-4) if the latter limit exists.

1.3 Proof of Campbell's Theorem

Consider the case in which exactly K electrons arrive at the anode in an interval of length T. Before the interval starts, we think of these K electrons as fated to arrive in the interval $(0, T)$ but any particular electron is just as likely to arrive at one time as any other time. We shall number these fated electrons from one to K for purposes of identification but it is to be emphasized that the numbering has nothing to do with the order of arrival. Thus, if t_k be the time of arrival of electron number k, the probability that t_k lies in the interval $(t, t + dt)$ is dt/T.

We take T to be very large compared with the range of values of t for which $F(t)$ is appreciably different from zero. In physical applications such a range usually exists and we shall call it Δ even though it is not very definite. Then, when exactly K electrons arrive in the interval $(0, T)$ the effect is approximately

$$
I_K(t) = \sum_{k=1}^{K} F(t - t_k) \qquad (1.3\text{-}1)
$$

the degree of approximation being very good over all of the interval except within Δ of the end points.

Suppose we examine a large number M of intervals of length T. The number having exactly K arrivals will be, to a first approximation $M\ p(K)$ where $p(K)$ is given by (1.1-3). For a fixed value of t and for each interval having K arrivals, $I_K(t)$ will have a definite value. As $M \to \infty$, the average value of the $I_K(t)$'s, obtained by averaging over the intervals, is

$$
\begin{aligned}
\bar{I}_K(t) &= \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T} \sum_{k=1}^{K} F(t - t_k) \\
&= \sum_{k=1}^{K} \int_0^T \frac{dt_k}{T} F(t - t_k)
\end{aligned}
\qquad (1.3\text{-}2)
$$

and if $\Delta < t < T - \Delta$, we have effectively

$$\bar{I}_K(t) = \frac{K}{T}\int_{-\infty}^{+\infty} F(t)\,dt \tag{1.3-3}$$

If we now average $I(t)$ over all of the M intervals instead of only over those having K arrivals, we get, as $M \to \infty$,

$$\begin{aligned}\bar{I}(t) &= \sum_{K=0}^{\infty} p(K)\bar{I}_K(t) \\ &= \sum_{K=0}^{\infty} \frac{K}{T}\frac{(\nu T)^K}{K!}e^{-\nu T}\int_{-\infty}^{+\infty} F(t)\,dt \\ &= \nu\int_{-\infty}^{+\infty} F(t)\,dt \end{aligned} \tag{1.3-4}$$

and this proves the first part of the theorem. We have used this rather elaborate proof to prove the relatively simple (1.3-4) in order to illustrate a method which may be used to prove more complicated results. Of course, (1.3-4) could be established by noting that the integral is the average value of the effect produced by one arrival, the average being taken over one second, and that ν is the average number of arrivals per second.

In order to prove the second part, (1.2-3) of Campbell's theorem we first compute $\overline{I^2(t)}$ and use

$$\begin{aligned}\overline{(I(t) - \bar{I}(t))^2} &= \overline{I^2(t)} - 2\,\overline{I(t)\bar{I}(t)} + \bar{I}(t)^2 \\ &= \overline{I^2(t)} - \bar{I}(t)^2\end{aligned} \tag{1.3-5}$$

From the definition (1.3-1) of $I_K(t)$,

$$I_K^2(t) = \sum_{k=1}^{K}\sum_{m=1}^{K} F(t - t_k)F(t - t_m)$$

Averaging this over all values of $t_1, t_2, \cdots t_K$ with t held fixed as in (1.3-2),

$$\overline{I_K^2(t)} = \sum_{k=1}^{K}\sum_{m=1}^{K}\int_0^T \frac{dt_1}{T}\cdots\int_0^T \frac{dt_K}{T} F(t - t_k)F(t - t_m)$$

The multiple integral has two different values. If $k = m$ its value is

$$\int_0^T F^2(t - t_k)\,\frac{dt_k}{T}$$

and if $k \neq m$ its value is

$$\int_0^T F(t - t_k)\,\frac{dt_k}{T}\int_0^T F(t - t_m)\,\frac{dt_m}{T}$$

Counting up the number of terms in the double sum shows that there are K of them having the first value and $K^2 - K$ having the second value. Hence, if $\Delta < t < T - \Delta$ we have

$$I_K^2(t) = \frac{K}{T}\int_{-\infty}^{+\infty} F^2(t)\,dt + \frac{K(K-1)}{T^2}\left[\int_{-\infty}^{+\infty} F(t)\,dt\right]^2$$

Averaging over all the intervals instead of only those having K arrivals gives

$$\overline{I^2(t)} = \sum_{K=0}^{\infty} p(K)\, \overline{I_K^2(t)}$$

$$= \nu \int_{-\infty}^{+\infty} F^2(t)\,dt + \bar{I}(t)^2$$

where the summation with respect to K is performed as in (1.3-4), and after summation the value (1.3-4) for $\bar{I}(t)$ is used. Comparison with (1.3-5) establishes the second part of Campbell's theorem.

1.4 The Distribution of $I(t)$

When certain conditions are satisfied the proportion of time which $I(t)$ spends in the range $I, I + dI$ is $P(I)dI$ where, as $\nu \to \infty$, the probability density $P(I)$ approaches

$$\frac{1}{\sigma_I\sqrt{2\pi}}\, e^{-(I-\bar{I})^2/2\sigma_I^2} \qquad (1.4\text{-}1)$$

where $\bar{I}$ is the average of $I(t)$ given by (1.2-2) and the square of the standard deviation σ_I, i.e. the variance of $I(t)$, is given by (1.2-3). This normal distribution is the one which would be expected by virtue of the "central limit theorem" in probability. This states that, under suitable conditions, the distribution of the sum of a large number of random variables tends toward a normal distribution whose variance is the sum of the variances of the individual variables. Similarly the average of the normal distribution is the sum of the averages of the individual variables.

So far, we have been speaking of the limiting form of the probability density $P(I)$. It is possible to write down an explicit expression for $P(I)$, which, however, is quite involved. From this expression the limiting form may be obtained. We now obtain this expression. In line with the discussion given of Campbell's theorem, we seek the probability density $P(I)$ of the values of $I(t)$ observed at t seconds from the beginning of each of a large number, M, of intervals, each of length T.

Probability that $I(t)$ lies in range $(I, I + dI)$

$$= \sum_{K=0}^{\infty} \text{(Probability of exactly } K \text{ arrivals)} \times$$

$$\text{(Probability that if there are exactly } K \text{ arrivals, } I_K(t) \text{ lies in } (I, I + dI)).$$

Denoting the last probability in the summation by $P_K(I)dI$, using notation introduced earlier, and cancelling out the factor dI gives

$$P(I) = \sum_{K=0}^{\infty} p(K)P_K(I) \tag{1.4-2}$$

We shall compute $P_K(I)$ by the method of "characteristic functions"[8] from the definition

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.3-1}$$

of $I_K(t)$. The method will be used in its simplest form: the probability that the sum

$$x_1 + x_2 + \cdots + x_K$$

of K independent random variables lies between X and $X + dX$ is

$$dX \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iXu} \prod_{k=1}^{K} (\text{average value of } e^{ix_k u})\, du \tag{1.4-3}$$

The average value of $e^{ix_k u}$, i.e., the characteristic function of the distribution of x_k, is obtained by averaging over the values of x_k. Although this is the simplest form of the method it is also the least general in that the integral does not converge for some important cases. The distribution which gives a probability of $\frac{1}{2}$ that $x_k = -1$ and $\frac{1}{2}$ that $x_k = +1$ is an example of such a case. However, we may still use (1.4-3) formally in such cases by employing the relation

$$\int_{-\infty}^{+\infty} e^{-iau}\, du = 2\pi\delta(a) \tag{1.4-4}$$

where $\delta(a)$ is zero except at $a = 0$ where it is infinite and its integral from $a = -\epsilon$ to $a = +\epsilon$ is unity where $\epsilon > 0$.

When we identify x_k with $F(t - t_k)$ we see that the average value of $e^{ix_k u}$ is

$$\frac{1}{T} \int_0^T \exp\,[iuF(t - t_k)]\, dt_k$$

[8] The essentials of this method are due to Laplace. A few remarks on its history are given by E. C. Molina, *Bull. Amer. Math. Soc.*, 36 (1930), pp. 369–392. An account of the method may be found in any one of several texts on probability theory. We mention "Random Variables and Probability Distributions," by H. Cramér, Camb. Tract in Math. and Math. Phys. No. 36 (1937), Chap. IV. Also "Introduction to Mathematical Probability," by J. V. Uspensky, McGraw-Hill (1937), pages 240, 264, and 271–278.

All of the K characteristic functions are the same and hence, from (1.4-3), $P_K(I)dI$ is

$$dI \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iIu} \left(\frac{1}{T} \int_0^T \exp [iuF(t - \tau)] \, d\tau \right)^K du$$

Although in deriving this relation we have taken $K > 0$, it also holds for $K = 0$ (provided we use (1.4-4)). In this case $P_0(I) = \delta(I)$, because $I = 0$ when no electrons arrive.

Inserting our expression for $P_K(I)$ and the expression (1.1-3) for $p(K)$ in (1.4-2) and performing the summation gives

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left(-iIu - \nu T + \nu \int_0^T \exp [iuF(t - \tau)] \, d\tau \right) du \qquad (1.4\text{-}5)$$

The first exponential may be simplified somewhat. Using

$$\nu T = \nu \int_0^T d\tau$$

permits us to write

$$-\nu T + \nu \int_0^T \exp [iuF(t - \tau)] \, d\tau = \nu \int_0^T (\exp [iuF(t - \tau)] - 1) \, d\tau$$

Suppose that $\Delta < t < T - \Delta$ where Δ is the range discussed in connection with equation (1.3-1). Taking $| F(t - \tau) | = 0$ for $| t - \tau | > \Delta$ then enables us to write the last expression as

$$\nu \int_{-\infty}^{+\infty} [e^{iuF(t)} - 1] \, dt \qquad (1.4\text{-}6)$$

Placing this in (1.4-5) yields the required expression for $P(I)$:

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left(-iIu + \nu \int_{-\infty}^{+\infty} [e^{iuF(t)} - 1] \, dt \right) du \qquad (1.4\text{-}7)$$

An idea of the conditions under which the normal law (1.4-1) is approached may be obtained from (1.4-7) by expanding (1.4-6) in powers of u and determining when the terms involving u^3 and higher powers of u may be neglected. This is taken up for a slightly more general form of current in section 1.6.

1.5 Extension of Campbell's Theorem

In section 1.2 we have stated Campbell's theorem. Here we shall give an extension of it. In place of the expression (1.2-1) for the $I(t)$ of the shot effect we shall deal with the current

$$I(t) = \sum_{k=-\infty}^{+\infty} a_k F(t - t_k) \tag{1.5-1}$$

where $F(t)$ is the same sort of function as before and where $\cdots a_1, a_2, \cdots a_k, \cdots$ are independent random variables all having the same distribution. It is assumed that all of the moments $\overline{a^n}$ exist, and that the events occur at random

The extension states that the nth semi-invariant of the probability density $P(I)$ of I, where I is given by (1.5-1), is

$$\lambda_n = \nu \overline{a^n} \int_{-\infty}^{+\infty} [F(t)]^n \, dt \tag{1.5-2}$$

where ν is the expected number of events per second. The semi-invariants of a distribution are defined as the coefficients in the expansion

$$\log_e (\text{ave. } e^{iIu}) = \sum_{n=1}^{N} \frac{\lambda_n}{n!} (iu)^n + o(u^N) \tag{1.5-3}$$

i.e. as the coefficients in the expansion of the logarithm of the characteristic function. The λ's are related to the moments of the distribution. Thus if $m_1, m_2, \cdots$ denote the first, second $\cdots$ moments about zero we have

$$\text{ave. } e^{iIu} = 1 + \sum_{n=1}^{N} \frac{m_n}{n!} (iu)^n + o(u^N)$$

By combining this relation with the one defining the λ's it may be shown that

$$\begin{aligned} \bar{I} &= m_1 = \lambda_1 \\ \overline{I^2} &= m_2 = \lambda_2 + \lambda_1 m_1 \\ \overline{I^3} &= m_3 = \lambda_3 + 2\lambda_2 m_1 + \lambda_1 m_2 \\ &\cdots\cdots\cdots \end{aligned}$$

It follows that $\lambda_1 = \bar{I}$ and $\lambda_2 = \text{ave. } (I - \bar{I})^2$. Hence (1.5-2) yields the original statement of Campbell's theorem when we set n equal to one and two and also take all the a's to be unity.

The extension follows almost at once from the generalization of expression (1.4-7) for the probability density $P(I)$. By proceeding as in section 1.4 and identifying x_k with $a_k F(t - t_k)$ we see that

$$\text{ave. } e^{ix_k u} = \frac{1}{T} \int_{-\infty}^{+\infty} q(a) \, da \int_0^T \exp [iuaF(t - t_k)] \, dt_k$$

where $q(a)$ is the probability density function for the a's. It turns out that the probability density $P(I)$ of I as defined by (1.5-1) is

$$P(I) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-iIu + \nu\int_{-\infty}^{+\infty} q(a)\,da \int_{-\infty}^{+\infty} [e^{iuaF(t)} - 1]\,dt\right) du \qquad (1.5\text{-}4)$$

The logarithm of the characteristic function of $P(I)$ is, from (1.5-4),

$$\nu\int_{-\infty}^{+\infty} q(a)\,da \int_{-\infty}^{+\infty} [e^{iuaF(t)} - 1]\,dt = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!}\,\nu\int_{-\infty}^{+\infty} q(a)\,da\,a^n \int_{-\infty}^{+\infty} F^n(t)\,dt$$

Comparison with the series (1.5-3) defining the semi-invariants gives the extension of Campbell's theorem stated by (1.5-2).

Other extensions of Campbell's theorem may be made. For example, suppose in the expression (1.5-1) for $I(t)$ that $t_1, t_2, \cdots t_k, \cdots$ while still random variables, are no longer necessarily distributed according to the laws assumed above. Suppose now that the probability density $p(x)$ is given where x is the interval between two successive events:

$$t_2 = t_1 + x_1 \qquad (1.5\text{-}5)$$

$$t_3 = t_2 + x_2 = t_1 + x_1 + x_2$$

and so on. For the case treated above

$$p(x) = \nu e^{-\nu x} \qquad (1.5\text{-}6)$$

We assume that the expected number of events per second is still ν.

Also we take the special, but important, case for which

$$F(t) = 0, \qquad t < 0 \qquad (1.5\text{-}7)$$

$$F(t) = e^{-\alpha t}, \qquad t > 0.$$

For a very long interval extending from $t = t_1$ to $t = T + t_1$ inside of which there are exactly K events we have, if t is not near the ends of the interval,

$$I(t) = a_1F(t - t_1) + a_2F(t - t_1 - x_1) + \cdots + a_{K+1}F(t - t_1 - x_1 - \cdots - x_K)$$

$$= a_1F(t') + a_2F(t' - x_1) + \cdots + a_{K+1}F(t' - x_1 - \cdots - x_K)$$

$$I^2(t) = a_1^2F^2(t') + a_2^2F^2(t' - x_1) + \cdots + a_{K+1}^2F^2(t' - x_1 \cdots - x_K)$$
$$+ 2a_1a_2F(t')F(t' - x_1) + \cdots + 2a_1a_{K+1}F(t')F(t' - x_1 \cdots - x_K)$$
$$+ 2a_2a_3F(t' - x_1)F(t' - x_1 - x_2) + \cdots + \cdots$$

where $t' = t - t_1$. If we integrate $I^2(t)$ over the entire interval $0 < t' < T$ and drop the primes we get approximately

$$\int_0^T I^2(t)dt = (a_1^2 + \cdots + a_{K+1}^2)\varphi(0)$$
$$+ 2a_1a_2\varphi(x_1) + 2a_1a_3\varphi(x_1 + x_2) + \cdots + 2a_1a_{K+1}\varphi(x_1 + \cdots + x_K)$$
$$+ 2a_2a_3\varphi(x_2) + \cdots + \cdots + 2a_Ka_{K+1}\varphi(x_K)$$

where

$$\varphi(x) = \int_{-\infty}^{+\infty} F(t)F(t - x)\,dx$$

When we divide both sides by T and consider K and T to be very large,

$$\frac{K}{T}\,\frac{a_1^2 + \cdots + a_{K+1}^2}{K}\,\varphi(0) = \nu a^2\varphi(0)$$

$$\frac{1}{T}[a_1a_2\,\varphi(x_1) + a_2a_3\,\varphi(x_2) + \cdots + a_K\,a_{K+1}\,\varphi(x_K)] = \frac{K}{T}\text{ average } a_k\,a_{k+1}\,\varphi(x_k)$$
$$= \nu\bar{a}^2\int_0^\infty \varphi(x)p(x)\,dx$$

$$\frac{1}{T}[a_1\,a_3\,\varphi(x_1 + x_2) + \cdots] = \frac{K-1}{T}\text{ ave. } a_k\,a_{k+3}\,\varphi(x_k + x_{k+1})$$
$$= \nu\bar{a}^2\int_0^\infty dx_1\int_0^\infty dx_2\,p(x_1)p(x_2)\varphi(x_1 + x_2)$$

Consequently

$$\overline{I^2(t)} = \lim_{T\to\infty}\frac{1}{T}\int_0^T I^2(t)\,dt$$
$$= \nu a^2\varphi(0) + 2\nu\bar{a}^2\Bigg[\int_0^\infty p(x)\varphi(x)\,dx$$
$$+ \int_0^\infty dx_1\int_0^\infty dx_2\,p(x_1)p(x_2)\varphi(x_1 + x_2) + \cdots\Bigg]$$

For our special exponential form (1.5 7) for $F(t)$,

$$\varphi(x) = \frac{\epsilon^{-\alpha x}}{2\alpha}$$

and the multiple integrals occurring in the expression for $\overline{I^2(t)}$ may be written in terms of powers of

$$q = \int_0^\infty p(x) e^{-\alpha x} \, dx \tag{1.5-8}$$

Thus

$$2\alpha \overline{I^2(t)} = \nu \overline{a^2} + 2\bar{a}^2 \nu \frac{q}{1 - q}$$

and since

$$\bar{I}(t) = \nu \bar{a} \int_{-\infty}^{+\infty} F(t) \, dt = \nu \bar{a}/\alpha$$

we have

$$\overline{I^2(t)} - \overline{I(t)}^2 = \frac{\nu \overline{a^2}}{2\alpha} + \left(\frac{\nu \bar{a}}{\alpha}\right)^2 \left[\frac{\alpha q}{\nu(1 - q)} - 1\right] \tag{1.5-9}$$

Equations (1.5-8) and (1.5-9) give us an extension of Campbell's theorem subject to the restrictions discussed in connection with equations (1.5-5) and (1.5-7). Other generalizations have been made[4] but we shall leave the subject here. The reader may find it interesting to verify that (1.5-9) gives the correct answer when $p(x)$ is given by (1.5-6), and also to investigate the case when the events are spaced equally.

1.6 Approach of Distribution of I to a Normal Law

In section 1.5 we saw that the probability density $P(I)$ of the noise current I may be expressed formally as

$$P(I) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[-iIu + \sum_{n=1}^{\infty} (iu)^n \lambda_n / n!\right] du \tag{1.6-1}$$

where λ_n is the nth semi-invariant given by (1.5-2). By setting

$$\lambda_2 = \sigma^2$$

$$x = \frac{I - \lambda_1}{\sigma} = \frac{I - \bar{I}}{\sigma} \tag{1.6-2}$$

[4] See E. N. Rowland, *Proc. Camb. Phil. Soc.* 32 (1936), 580-597. He extends the theorem to the case where there are two functions instead of a single one, which we here denote by $I(t)$. According to a review in the Zentralblatt für Math., 19, p. 224, Khintchine in the Bull. Acad. Sci. URSS, sér. Math. Nr. 3 (1938), 313-322, has continued and made precise the earlier work of Rowland.

expanding

$$\exp \sum_{n=3}^{\infty} (iu)^n \lambda_n / n!$$

as a power series in u, integrating termwise using

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} (iu\sigma)^n \exp\left[-iu\sigma x - \frac{u^2\sigma^2}{2}\right] du = (-)^n \sigma^{-1} \varphi^{(n)}(x),$$

$$\varphi^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dx^n} e^{-x^2/2}$$

and finally collecting terms according to their order in powers of $\nu^{-1/2}$, gives

$$P(I) \sim \sigma^{-1}\varphi^{(0)}(x) - \frac{\lambda_3 \sigma^{-4}}{3!}\varphi^{(3)}(x) + \left[\frac{\lambda_4\sigma^{-5}}{4!}\varphi^{(4)}(x) + \frac{\lambda_3^2\sigma^{-7}}{72}\varphi^{(6)}(x)\right] + \cdots \tag{1.6-3}$$

The first term is $O(\nu^{-1/2})$, the second term is $O(\nu^{-1})$, and the term within brackets is $O(\nu^{-3/2})$. This is Edgeworth's series.[5] The first term gives the normal distribution and the remaining terms show how this distribution is approached as $\nu \to \infty$.

1.7 The Fourier Components of $I(t)$

In some analytical work noise current is represented as

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right) \tag{1.7-1}$$

where at a suitable place in the work T and N are allowed to become infinite. The coefficients a_n and b_n, $1 \leq n \leq N$, are regarded as independent random variables distributed about zero according to a normal law.

It appears that the association of (1.7-1) with a sequence of disturbances occurring at random goes back many years. Rayleigh[6] and Gouy suggested that black-body radiation and white light might both be regarded as sequences of irregularly distributed impulses.

Einstein[7] and von Laue have discussed the normal distribution of the coefficients in (1.7-1) when it is used to represent black-body radiation, this radiation being the resultant produced by a great many independent os-

[5] See, for example, pp. 86-87, in "Random Variables and Probability Distributions" by H. Cramér, *Cambridge Tract* No. 36 (1937).

[6] *Phil. Mag.* Ser. 5, Vol. 27 (1889) pp. 460-469.

[7] A. Einstein and L. Hopf, *Ann. d. Physik* 33 (1910) pp. 1095-1115.
M. V. Laue, *Ann. d. Physik* 47 (1915) pp. 853-878.
A. Einstein, *Ann. d. Physik* 47 (1915) pp. 879-885.
M. V. Laue, *Ann. d. Physik* 48 (1915) pp. 668-680.
I am indebted to Prof. Goudsmit for these references.

cillators. Some argument arose as to whether the coefficients in (1.7-1) were statistically independent or not. It was finally decided that they are independent.

The shot effect current has been represented in this way by Schottky.[8] The Fourier series representation has been discussed by H. Nyquist[9] and also by Goudsmit and Weiss. Remarks made by A. Schuster[10] are equivalent to the statement that a_n and b_n are distributed normally.

In view of this wealth of information on the subject it may appear superfluous to say anything about it. However, for the sake of completeness, we shall outline the thoughts which lead to (1.7-1).

In line with our usual approach to the shot effect, we suppose that exactly K electrons arrive during the interval $(0, T)$, so that the noise current for the interval is

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.7-2}$$

The coefficients in the Fourier series expansion of $I_K(t)$ over the interval $(0, T)$ are a_{nK} and b_{nK} where

$$\begin{aligned} a_{nK} - ib_{nK} &= \frac{2}{T} \sum_{k=1}^{K} \int_0^T F(t - t_k) \exp\left[-i \frac{2\pi nt}{T} \right] dt \\ &= \frac{2}{T} \sum_{k=1}^{K} \int_{-\infty}^{+\infty} F(t) \exp\left[-i \frac{2\pi n}{T} (t + t_k) \right] dt \\ &= R_n e^{-i\varphi_n} \sum_{k=1}^{K} e^{-in\theta_k} \end{aligned} \tag{1.7-3}$$

In this expression

$$\begin{aligned} \theta_k &= \frac{2\pi t_k}{T} \\ R_n e^{-i\varphi_n} &= C_n - iS_n = \frac{2}{T} \int_{-\infty}^{+\infty} F(t) e^{-i2\pi nt/T}\, dt \end{aligned} \tag{1.7-4}$$

In the earlier sections the arrival times $t_1, t_2, \cdots t_K$ were regarded as K independent random variable each distributed uniformly over the interval $(0, T)$. Hence the θ_k's may be regarded as random variables distributed uniformly over the interval 0 to 2π.

Incidentally, it will be noted that in (1.7-3) there occurs the sum of K randomly oriented unit vectors. When K becomes very large, as it does

[8] *Ann. d. Physik*, 57 (1918) pp. 541-567.

[9] Unpublished Memorandum, "Fluctuations in Vacuum Tube Noise and the Like," March 17, 1932.

[10] Investigation of Hidden Periodicities, Terrestrial Magnetism, 3 (1898), pp. 13-41. See especially propositions 1 and 2 on page 26 of Schuster's paper.

when $\nu \to \infty$, it is known that the real and imaginary parts of this sum are random variables, which tend to become independent and normally distributed about zero. This suggests the manner in which the normal distribution of the coefficients arises. Averaging over the θ_k's in (1.7-3) gives when $n > 0$

$$\bar{a}_{nK} = \bar{b}_{nK} = 0 \tag{1.7-5}$$

Some further algebra gives

$$\begin{aligned} \overline{a^2_{nK}} &= \overline{b^2_{nK}} = \frac{K}{2} R^2_n \\ \overline{a_{nK} b_{nK}} &= \overline{a_{nK} a_{mK}} = \overline{b_{nK} b_{mK}} = 0 \end{aligned} \tag{1.7-6}$$

where $n \neq m$ and $n, m > 0$.

So far, we have been considering the case of exactly K arrivals in our interval of length T. Now we pass to the general case of any number of arrivals by making use of formulas analogous to

$$\overline{a^2_n} = \sum_{K=0}^{\infty} p(K) \overline{a^2_{nK}} \tag{1.7-7}$$

as has been done in section 1.3. Thus, for $n > 0$,

$$\begin{aligned} \bar{a}_n &= \bar{b}_n = 0 \\ \overline{a^2_n} &= \overline{b^2_n} = \frac{\nu T}{2} R^2_n = \sigma^2_n \\ \overline{a_n b_n} &= \overline{a_n a_m} = \overline{b_n b_m} = 0, \qquad n \neq m \end{aligned} \tag{1.7-8}$$

In the second line we have used σ_n to denote the standard deviation of a_n and b_n. We may put the expression for σ^2_n in a somewhat different form by writing

$$f_n = \frac{n}{T} = n\Delta f, \qquad \Delta f = \frac{1}{T} \tag{1.7-9}$$

where f_n is the frequency of the nth component. Using (1.7-4),

$$\sigma^2_n = 2\nu\Delta f \left| \int_{-\infty}^{+\infty} F(t) e^{-i2\pi f_n t} \, dt \right|^2 \tag{1.7-10}$$

Thus, σ^2_n is proportional to ν/T.

The probability density function $P(a_1, \cdots a_N, b_1, \cdots b_N)$ for the $2N$ coefficients, $a_1, \cdots a_N, b_1, \cdots b_N$ may be derived in much the same fashion as was the probability density of the noise current in section 1.4. Here N

is arbitrary but fixed. The expression analogous to (1.4-5) is the $2N$ fold integral

$$P(a_1, \cdots, b_N) = (2\pi)^{-2N} \int_{-\infty}^{+\infty} du_1 \cdots \int_{-\infty}^{+\infty} dv_N \qquad (1.7\text{-}11)$$
$$\exp\left[-i(a_1 u_1 + \cdots + b_N v_N) - \nu T + \nu T E\right]$$

where

$$E = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp\left[i \sum_{n=1}^{N} (u_n C_n + v_n S_n) \cos n\theta + (v_n C_n - u_n S_n) \sin n\theta \right] \qquad (1.7\text{-}12)$$

in which $C_n - iS_n$ is defined as the Fourier transform (1.7-4) of $F(t)$.

The next step is to show that (1.7-11) approaches a normal law in $2N$ dimensions as $\nu \to \infty$. This appears to be quite involved. It will be noted that the integrand in the integral defining E is composed of N factors of the form

$$\exp\left[i\rho_n \cos (n\theta - \psi_n)\right]$$
$$= J_0(\rho_n) + 2i \cos (n\theta - \psi_n) J_1(\rho_n) - 2 \cos (2n\theta - 2\psi_n) J_2(\rho_n) + \cdots$$

where

$$\rho_n^2 = (u_n^2 + v_n^2)(C_n^2 + S_n^2) = \frac{2}{\nu T} \sigma_n^2 (u_n^2 + v_n^2).$$

As ν becomes large, it turns out that the integral (1.7-11) for the probability density obtains most of its contributions from small values of u and v. By substituting the product of the Bessel function series in the integral for E and integrating we find

$$E = \prod_{n=1}^{N} J_0(\rho_n) + A + B + C$$

where A is the sum of products such as

$$-2i \cos (\psi_{k+l} - \psi_k - \psi_l) J_1(\rho_k) J_1(\rho_l) J_1(\rho_{k+l}) \text{ times } N - 3\ J_0\text{'s}$$

in which $0 < k \leq l$ and $2 \leq k + l \leq N$. Similarly B is the sum of products of the form

$$-2i \cos (\psi_{2k} - 2\psi_k) J_1(\rho_{2k}) J_2(\rho_k) \text{ times } N - 2\ J_0\text{'s}$$

C consists of terms which give fourth and higher powers in u and v. There are roughly $N^2/4$ terms of form A and $N/2$ terms of form B.

Expanding the Bessel functions, neglecting all powers above the third and proceeding as in section 1.4, will give us the normal distribution plus the first correction term. It is rather a messy affair. An idea of how it looks may be obtained by taking the special case in which $F(t)$ is an even function of t and neglecting terms of type B. Then

$$P(a_1, \cdots a_N, b_1, \cdots b_N) = (1 + \eta) \prod_{n=1}^{N} \frac{e^{-(x_n^2+y_n^2)/2}}{2\pi\sigma_n^2} \qquad (1.7\text{-}12)$$

where

$$x_n = \frac{a_n}{\sigma_n}, \qquad y_n = \frac{b_n}{\sigma_n}$$

$$\eta = (2\nu T)^{-1/2} \sum_{k,\ell} [x_{k+\ell}(x_k x_\ell - y_k y_\ell) + 2\, y_{k+\ell}\, y_k y_\ell] \qquad (1.7\text{-}13)$$

and the summation extends over $2 \leq k + l \leq N$ with $k \leq l$.

It is seen that if T and N are held constant, the correction term η approaches zero as ν becomes very large. A very rough idea of the magnitude of η may be obtained by assuming that unity is a representative value of the x's and y's. Further assuming that there are N^2 terms in the summation each one of which may be positive or negative suggests that magnitude of the sum is of the order of N. Hence we might expect to find that η is of the order of $N(2\nu T)^{-1/2}$.

PART II

POWER SPECTRA AND CORRELATION FUNCTIONS

2.0 INTRODUCTION

A theory for analyzing functions of time, t, which do not die down and which remain finite as t approaches infinity has gradually been developed over the last sixty years. A few words of its history together with an extensive bibliography are given by N. Wiener in his paper on "Generalized Harmonic Analysis".[11] G. Gouy, Lord Rayleigh and A. Schuster were led to study this problem in their investigations of such things as white light and noise. Schuster[12] invented the "periodogram" method of analysis which has as its object the discovery of any periodicities hidden in a continuous curve representing meteorological or economic data.

[11] *Acta Math.*, Vol. 55, pp. 117-258 (1930). See also "Harmonic Analysis of Irregular Motion," *Jour. Math. and Phys.* 5 (1926) pp. 99-189.

[12] The periodogram was first introduced by Schuster in reference 10 cited in Section 1.7. He later modified its definition in the Trans. Camb. Phil. Soc. 18 (1900), pp. 107-135, and still later redefined it in "The Periodogram and its Optical Analogy," Proc. Roy. Soc., London, Ser. A, 77 (1906) pp. 136-140. In its final form the periodogram is equivalent to $\frac{1}{4}w(f)$, where $w(f)$ is the power spectrum defined in Section 2.1, plotted as a function of the period $T = (2\pi f)^{-1}$.

The correlation function, which turns out to be a very useful tool, apparently was introduced by G. I. Taylor.[13] Recently it has been used by quite a few writers[14] in the mathematical theory of turbulence.

In section 2.1 the power spectrum and correlation function of a specific function, such as one given by a curve extending to $t = \infty$, are defined by equations (2.1-3) and (2.1-4) respectively. That they are related by the Fourier inversion formulae (2.1-5) and (2.1-6) is merely stated; the discussion of the method of proof being delayed until sections 2.3 and 2.4. In section 2.3 a discussion based on Fourier series is given and in section 2.4 a parallel treatment starting with Parseval's integral theorem is set forth. The results as given in section 2.1 have to be supplemented when the function being analyzed contains a d.c. or periodic components. This is taken up in section 2.2.

The first four sections deal with the analysis of a specific function of t. However, most of the applications are made to functions which behave as though they are more or less random in character. In the mathematical analysis this randomness is introduced by assuming the function of t to be also a function of suitable parameters, and then letting these parameters be random variables. This question is taken up in section 2.5. In section 2.6 the results of 2.5 are applied to determine the average power spectrum and the average correlation function of the shot effect current. The same thing is done in 2.7 for a flat top wave, the tops (and bottoms) being of random length. The case in which the intervals are of equal length but the sign of the wave is random is also discussed in 2.7. The representation of the noise current as a trigonometrical series with random variable coefficients is taken up in 2.8. The last two sections 2.9 and 2.10 are devoted to probability theory. The normal law and the central limit theorem, respectively, are discussed.

2.1 Some Results of Generalized Harmonic Analysis

We shall first state the results which we need, and then show that they are plausible by methods which are heuristic rather than rigorous. Suppose that $I(t)$ is one of the functions mentioned above. We may think of it as being specified by a curve extending from $t = -\infty$ to $t = \infty$. $I(t)$ may be regarded as composed of a great number of sinusoidal components whose frequencies range from 0 to $+\infty$. It does not necessarily have to be a noise current, but if we think of it as such, then, in flowing through a resistance of one ohm it will dissipate a certain average amount of power, say p watts.

[13] Diffusion by Continuous Movements, *Proc. Lond. Math. Soc.*, Ser. 2, 20, pp. 196–212 (1920).

[14] See the text "Modern Developments in Fluid Dynamics" edited by S. Goldstein, Oxford (1938).

That portion of ρ arising from the components having frequencies between f and $f + df$ will be denoted by $w(f)df$, and consequently

$$\rho = \int_0^\infty w(f)df \tag{2.1-1}$$

Since $w(f)$ is the spectrum of the average power we shall call it the "power spectrum" of $I(t)$. It has the dimensions of energy and on this account is frequently called the "energy-frequency spectrum" of $I(t)$. A mathematical formulation of this discussion leads to a clear cut definition of $w(f)$.

Let $\Phi(t)$ be a function of t, which is zero outside the interval $0 \leq t \leq T$ and is equal to $I(t)$ inside the interval. Its spectrum $S(f)$ is given by

$$S(f) = \int_0^T I(t)e^{-2\pi i f t}\, dt \tag{2.1-2}$$

The spectrum of the power, $w(f)$, is defined as

$$w(f) = \operatorname*{Limit}_{T\to\infty} \frac{2|S(f)|^2}{T} \tag{2.1-3}$$

where we consider only values of $f > 0$ and assume that this limit exists. This is substantially the definition of $w(f)$ given by J. R. Carson[15] and is useful when $I(t)$ has no periodic terms and no d.c. component. In the latter case (2.1-3) must either be supplemented by additional definitions or else a somewhat different method of approach used. These questions will be discussed in section 2.2.

The correlation function $\psi(\tau)$ of $I(t)$ is defined by the limit

$$\psi(\tau) = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I(t)I(t + \tau)\, dt \tag{2.1-4}$$

which is assumed to exist. $\psi(\tau)$ is closely related to the correlation coefficients used in statistical theory to measure the correlation of two random variables. In the present case the value of $I(t)$ at time t is one variable and its value at a different time $t + \tau$ is the other variable.

The spectrum of the power $w(f)$ and the correlation function $\psi(\tau)$ are related by the equations

$$w(f) = 4\int_0^\infty \psi(\tau) \cos 2\pi f\tau\, d\tau \tag{2.1-5}$$

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1-6}$$

[15] "The Statistical Energy-Frequency Spectrum of Random Disturbances," *B.S.T.J.*, Vol. 10, pp. 374–381 (1931).

It is seen that $\psi(\tau)$ is an even function of τ and that

$$\psi(0) = \rho \tag{2.1-7}$$

When either $\psi(\tau)$ or $w(f)$ is known the other may be obtained provided the corresponding integral converges.

2.2 Power Spectrum for D.C. and Periodic Components

As mentioned in section 2.1, when $I(t)$ has a d.c. or a periodic component the limit in the definition (2.1-3) for $w(f)$ does not exist for f equal to zero or to the frequency of the periodic component. Perhaps the most satisfactory method of overcoming this difficulty, from the mathematical point of view, is to deal with the integral of the power spectrum.[16]

$$\int_0^f w(g)\, dg \tag{2.2-1}$$

instead of with $w(f)$ itself.

The definition (2.1-4) for $\psi(\tau)$ still holds. If, for example,

$$I(t) = A + C \cos (2\pi f_0 t - \varphi) \tag{2.2-2}$$

$\psi(\tau)$ as given by (2.1-4) is

$$\psi(\tau) = A^2 + \frac{C^2}{2} \cos 2\pi f_0 \tau \tag{2.2-3}$$

The inversion formulas (2.1-5) and (2.1-6) give

$$\begin{aligned} \int_0^f w(g)\, dg &= \frac{2}{\pi} \int_0^\infty \psi(\tau) \frac{\sin 2\pi f \tau}{\tau}\, d\tau \\ \psi(\tau) &= \int_0^\infty \cos 2\pi f \tau\, d\left[\int_0^f w(g)\, dg\right] \end{aligned} \tag{2.2-4}$$

[16] This is done by Wiener,[11] loc. cit., and by G. W. Kenrick, "The Analysis of Irregular Motions with Applications to the Energy Frequency Spectrum of Static and of Telegraph Signals," *Phil. Mag.*, Ser. 7, Vol. 7, pp. 176-196 (Jan. 1929). Kenrick appears to be one of the first to apply, to noise problems, the correlation function method of computing the power spectrum (one of his problems is discussed in Sec. 2.7). He bases his work on results due to Wiener. Khintchine, in "Korrelationstheorie der stationären stochastischen Prozesse," *Math. Annalen*, 109 (1934), pp. 604-615, proves the following theorem: A necessary and sufficient condition that a function $R(t)$ may be the correlation function of a continuous, stationary, stochastic process is that $R(t)$ may be expressed as

$$R(t) = \int_{-\infty}^{+\infty} \cos tx\, dF(x)$$

where $F(x)$ is a certain distribution function. This expression for $R(t)$ is essentially the second of equations (2.2-4). Khintchine's work has been extended by H. Cramér, "On the theory of stationary random processes," *Ann. of Math.*, Ser. 2, Vol. 41 (1940), pp. 215-230. However, Khintchine and Cramér appear to be interested primarily in questions of existence, representation, etc., and do not stress the concept of the power spectrum.

where the last integral is to be regarded as a Stieltjes' integral. When the expression (2.2-3) for $\psi(\tau)$ is placed in the first formula of (2.2-4) we get

$$\int_0^f w(g)\,dg = \begin{cases} A^2 & \text{when } 0 < f < f_0 \\ A^2 + \dfrac{C^2}{2}, & \text{"} \quad f > f_0 \end{cases} \qquad (2.2\text{-}5)$$

When this expression is used in the second formula of (2.2-4), the increments of the differential are seen to be A^2 at $f = 0$ and $C^2/2$ at $f = f_0$. The resulting expression for $\psi(\tau)$ agrees with the original one.

Here we desire to use a less rigorous, but more convenient, method of dealing with periodic components. By examining the integral of $w(f)$ as given by (2.2-5) we are led to write

$$w(f) = 2A^2\delta(f) + \frac{C^2}{2}\delta(f - f_0)\cdot \qquad (2.2\text{-}6)$$

where $\delta(x)$ is an even unit impulse function so that if $\epsilon > 0$

$$\int_0^\epsilon \delta(x)\,dx = \frac{1}{2}\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = \tfrac{1}{2} \qquad (2.2\text{-}7)$$

and $\delta(x) = 0$ except at $x = 0$, where it is infinite. This enables us to use the simpler inversion formulas of section 2.1. The second of these, (2.1-6), is immediately seen to give the correct expression for $\psi(\tau)$. The first one, (2.1-5), gives the correct expression for $w(f)$ provided we interpret the integrals as follows:

$$\begin{aligned} \int_0^\infty \cos 2\pi f\tau\,d\tau &= \tfrac{1}{2}\delta(f) \\ \int_0^\infty \cos 2\pi f_0\tau \cos 2\pi f\tau\,d\tau &= \tfrac{1}{4}\delta(f - f_0) \end{aligned} \qquad (2.2\text{-}8)$$

It is not hard to show that these are in agreement with the fundamental interpretation

$$\int_{-\infty}^{+\infty} e^{-i2\pi ft}\,dt = \int_{-\infty}^{+\infty} e^{i2\pi ft}\,dt = \delta(f) \qquad (2.2\text{-}9)$$

which in its turn follows from a formal application of the Fourier integral formula and

$$\int_{-\infty}^{+\infty} \delta(f)e^{i2\pi ft}\,df = \int_{-\infty}^{+\infty} \delta(f)e^{-i2\pi ft}\,df = 1 \qquad (2.2\text{-}10)$$

We must remember that $f_0 > 0$ and $f \geq 0$ in (2.2-8) so that $\delta(f + f_0) = 0$ for $f \geq 0$.

The definition (2.1–3) for $w(f)$ gives the continuous part of the power spectrum. In order to get the part due to the d.c. and periodic components, which is exemplified by the expression (2.2–6) for $w(f)$ involving the δ functions, we must supplement (2.1–3) by adding terms of the type

$$2A^2\delta(f) + \frac{C^2}{2}\delta(f - f_0) = \left[\underset{T\to\infty}{\text{Limit}} \frac{2|S(0)|^2}{T^2}\right]\delta(f) + \left[\underset{T\to\infty}{\text{Limit}} \frac{2|S(f_0)|^2}{T^2}\right]\delta(f - f_0) \quad (2.2\text{–}11)$$

The correctness of this expression may be verified by calculating $S(f)$ for the $I(t)$ of this section given by (2.2–2), and actually carrying out the limiting process.

2.3 Discussion of Results of Section One—Fourier Series

The fact that the spectrum of power $w(f)$ and the correlation function $\psi(\tau)$ are related by Fourier inversion formulas is closely connected with Parseval's theorems for Fourier series and integrals. In this section we shall not use Parseval's theorems explicitly. We start with Fourier's series and use the concept of each component dissipating its share of energy independently of the behavior of the other components.

Let that portion of $I(t)$ which lies in the interval $0 \leq t < T$ be expanded in the Fourier series

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right) \quad (2.3\text{–}1)$$

where

$$a_n = \frac{2}{T}\int_0^T I(t)\cos\frac{2\pi nt}{T}\,dt$$
$$b_n = \frac{2}{T}\int_0^T I(t)\sin\frac{2\pi nt}{T}\,dt \quad (2.3\text{–}2)$$

Then for the interval $-\tau \leq t < T - \tau$,

$$I(t + \tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n(t + \tau)}{T} + b_n \sin\frac{2\pi n(t + \tau)}{T}\right) \quad (2.3\text{–}3)$$

Multiplying the series for $I(t)$ and $I(t + \tau)$ together and integrating with respect to t gives, after some reduction,

$$\frac{1}{T}\int_0^T I(t)I(t + \tau)\,dt = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n^2 + b_n^2)\cos\frac{2\pi n}{T}\tau + O\left(\frac{\tau I^2}{T}\right) \quad (2.3\text{–}4)$$

where the last term represents correction terms which must be added because the series (2.3-3) does not represent $I(t + \tau)$ in the interval $(T - \tau, T)$ when $\tau > 0$, or in the interval $(0, -\tau)$ if $\tau < 0$.

If $I(t)$ were a current and if it were to flow through one ohm for the interval $(0, T)$, each component would dissipate a certain average amount of power. The average power dissipated by the component of frequency $f_n = n/T$ cycles per second would be, from the Fourier series and elementary principles,

$$\begin{aligned} &\frac{1}{2}(a_n^2 + b_n^2) \text{ watts}, \qquad n \neq 0 \\ &\frac{a_0^2}{4} \qquad \text{watts}, \qquad n = 0 \end{aligned} \tag{2.3-5}$$

The band width associated with the nth component is the difference in frequency between the $n + 1$ th and nth components:

$$f_{n+1} - f_n = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T}\, cps$$

Hence if the average power in the band $f, f + df$ is defined as $w(f)df$, the average power in the band $f_{n+1} - f_n$ is

$$w(f_n)(f_{n+1} - f_n) = w\left(\frac{n}{T}\right)\frac{1}{T}$$

and, from (2.3-5), this is given by

$$\begin{aligned} w\left(\frac{n}{T}\right)\frac{1}{T} &= \frac{1}{2}(a_n^2 + b_n^2), \qquad n \neq 0 \\ w(0)\,\frac{1}{T} &= \frac{a_0^2}{4}, \qquad\qquad n = 0 \end{aligned} \tag{2.3-6}$$

When the coefficients in (2.3-4) are replaced by their expressions in terms of $w(f)$ we get

$$\begin{aligned} \frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt + O\left(\frac{\tau I^2}{T}\right) &= \frac{1}{T}\sum_{n=0}^{\infty} w\left(\frac{n}{T}\right)\cos\frac{2\pi n\tau}{T} \\ &= \int_0^{\infty} w\left(\frac{n}{T}\right)\cos\frac{2\pi n\tau}{T}\frac{dn}{T} \\ &= \int_0^{\infty} w(f)\cos 2\pi f\tau\, df \end{aligned} \tag{2.3-7}$$

where we have assumed T so large and $w(f)$ of such a nature that the summation may be replaced by integration.

If I remains finite, then as $T \to \infty$ with τ held fixed, the correction term on the left becomes negligibly small and we have, upon using the definitions (2.1-4) for the correlation function $\psi(\tau)$, the second of the fundamental inversion formulas (2.1-6). The first inversion formula may be obtained from this at once by using Fourier's double integral for $w(f)$.

Incidentally, the relation (2.3-6) between $w(f)$ and the coefficients a_n and b_n is in agreement with the definition (2.1-3) for $w(f)$ as a limit involving $|S(f)|^2$. From the expressions (2.3-2) for a_n and b_n, the spectrum $S(f_n)$ given by (2.1-2) is

$$S(f_n) = \frac{T}{2}(a_n - ib_n)$$

Then, from (2.1-3) $w(f_n)$ is given by the limit, as $T \to \infty$, of

$$\frac{2}{T}|S(f_n)|^2 = \frac{2}{T}\cdot\frac{T^2}{4}(a_n^2 + b_n^2)$$

$$= \frac{T}{2}(a_n^2 + b_n^2)$$

and this is the expression for $w\left(\frac{n}{T}\right)$ given by (2.3-6).

2.4 Discussion of Results of Section One—Parseval's Theorem

The use of Parseval's theorem[17] enables us to derive the results of section 2.1 more directly than the method of the preceding section. This theorem states that

$$\int_{-\infty}^{+\infty} F_1(f)F_2(f)\,df = \int_{-\infty}^{+\infty} G_1(t)G_2(-t)\,dt \qquad (2.4\text{-}1)$$

where F_1, G_1 and F_2, G_2 are Fourier mates related by

$$F(f) = \int_{-\infty}^{+\infty} G(t)e^{-i2\pi ft}\,dt$$

$$G(t) = \int_{-\infty}^{+\infty} F(f)e^{i2\pi ft}\,df \qquad (2.4\text{-}2)$$

It may be proved in a formal manner by replacing the F_1 on the left of (2.4-1) by its expression as an integral involving $G_1(t)$. Interchanging the

[17] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford (1937).

order of integration and using the second of (2.4–2) to replace F_2 by G_2 gives the right hand side.

We now set $G_1(t)$ and $G_2(t)$ equal to zero except for intervals of length T. These intervals and the corresponding values of G_1 and G_2 are

$$G_1(t) = I(t), \qquad 0 < t < T \tag{2.4–3}$$
$$G_2(t) = I(-t + \tau), \quad \tau - T < t < \tau$$

From (2.4–3) it follows that $F_1(f)$ is the spectrum $S(f)$ of $I(t)$ given by equation (2.1–2). Since $I(t)$ is real it follows from the first of equations (2.4–2) that

$$S(-f) = S^*(f), \tag{2.4–4}$$

where the star denotes conjugate complex, and hence that $|S(f)|^2$ is an even function of f.

The first of equations (2.4–2) also gives

$$\begin{aligned} F_2(f) &= \int_{\tau-T}^{\tau} I(-t + \tau)e^{-i2\pi ft}\,dt \\ &= \int_0^T I(t)e^{i2\pi f(t-\tau)}\,dt \\ &= S^*(f)e^{-i2\pi f\tau} \end{aligned} \tag{2.4–5}$$

When these G's and F's are placed in (2.4–1) we obtain

$$\int_{-\infty}^{+\infty} |S(f)|^2 e^{-2\pi f\tau}\,df = \int_0^{T-\tau} I(t)I(t+\tau)\,dt \tag{2.4–6}$$

where we have made use of the fact that $G_2(-t)$ is zero except in the interval $-\tau < t < T - \tau$ and have assumed $\tau > 0$. If $\tau < 0$ the limits of integration on the right would be $-\tau$ and T.

Since $|S(f)|^2$ is an even function of f we may write (2.4–6) as

$$\frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt + O\left(\frac{\tau I^2}{T}\right) = \int_0^{\infty} \frac{2|S(f)|^2}{T} \cos 2\pi f\tau\,df \tag{2.4–7}$$

If we now define the correlation function $\psi(\tau)$ as the limit, as $T \to \infty$, of the left hand side and define $w(f)$ as the function

$$w(f) = \operatorname*{Limit}_{T\to\infty} \frac{2|S(f)|^2}{T}, \qquad f > 0 \tag{2.4–3}$$

we obtain the second, (2.1–6), of the fundamental inversion formulas. As before, the first may be obtained from Fourier's integral theorem.

In order to obtain the interpretation of $w(f)df$ as the average power dissipated in one ohm by those components of $I(t)$ which lie in the band f, $f + df$, we set $\tau = 0$ in (2.4-7):

$$\underset{T\to\infty}{\text{Limit}} \frac{1}{T}\int_0^T I^2(t)\,dt = \int_0^\infty w(f)\,df \tag{2.4-8}$$

The expression on the left is certainly the total average power which would be dissipated in one ohm and the right hand side represents a summation over all frequencies extending from 0 to ∞. It is natural therefore to interpret $w(f)df$ as the power due to the components in $f, f + df$.

The preceding sections have dealt with the power spectrum $w(f)$ and correlation function $\psi(\tau)$ of a very general type of function. It will be noted that a knowledge of $w(f)$ does not enable us to determine the original function. In obtaining $w(f)$, as may be seen from the definition (2.1-3) or from (2.3-6), the information carried by the phase angles of the various components of $I(t)$ has been dropped out. In fact, as we may see from the Fourier series representation (2.3-1) of $I(t)$ and from (2.3-6), it is possible to obtain an infinite number of different functions all of which have the same $w(f)$, and hence the same $\psi(\tau)$. All we have to do is to assign different sets of values to the phase angles of the various components, thereby keeping $a_n^2 + b_n^2$ constant.

2.5 Harmonic Analysis for Random Functions

In many applications of the theory discussed in the foregoing sections $I(t)$ is a function of t which has a certain amount of randomness associated with it. For example $I(t)$ may be a curve representing the price of wheat over a long period of years, a component of air velocity behind a grid placed in a wind tunnel, or, of primary interest here, a noise current.

In some mathematical work this randomness is introduced by considering $I(t)$ to involve a number of parameters, and then taking the parameters to be random variables. Thus, in the shot effect the arrival times $t_1, t_2, \cdots t_K$ of the electrons were taken to be the parameters and each was assumed to be uniformly distributed over an interval $(0, T)$.

For any particular set of values of the parameters, $I(t)$ has a definite power spectrum $w(f)$ and correlation function $\psi(\tau)$. However, now the principal interest is not in these particular functions, but in functions which give the average values of $w(f)$ and $\psi(\tau)$ for fixed f and τ. These functions are obtained by averaging $w(f)$ and $\psi(\tau)$ over the ranges of the parameters, using, of course, the distribution functions of the parameters.

By averaging both sides of the appropriate equations in sections 2.1 and

2.2 it is seen that our fundamental inversion formulae (2.1-5) and (2.1-6) are unchanged. Thus,

$$\bar{w}(f) = 4\int_0^\infty \bar{\psi}(\tau)\cos 2\pi f\tau\, d\tau \tag{2.5-1}$$

$$\bar{\psi}(\tau) = \int_0^\infty \bar{w}(f)\cos 2\pi f\tau\, df \tag{2.5-2}$$

where the bars indicate averages taken over the parameters with f or τ held constant.

The definitions of $\bar{w}$ and $\bar{\psi}$ appearing in these equations are likewise obtained from (2.1-3) and (2.1-4)

$$\bar{w}(f) = \underset{T\to\infty}{\text{Limit}}\ \frac{2\overline{|S(f)|^2}}{T} \tag{2.5-3}$$

and

$$\bar{\psi}(\tau) = \underset{T\to\infty}{\text{Limit}}\ \frac{1}{T}\int_0^T \overline{I(t)I(t+\tau)}\, dt \tag{2.5-4}$$

The values of t and τ are held fixed while averaging over the parameters. In (2.5-3) $S(f)$ is regarded as a function of the parameters obtained from $I(t)$ by

$$S(f) = \int_0^T I(t)e^{-2\pi i f t}\, dt \tag{2.1-2}$$

Similar expressions may be obtained for the average power spectrum for d.c. and periodic components. All we need to do is to average the expression (2.2-11)

Sometimes the average value of the product $I(t)I(t+\tau)$ in the definition (2.5-4) of $\bar{\psi}(\tau)$ is independent of the time T. This enables us to perform the integration at once and obtain

$$\bar{\psi}(\tau) = \overline{I(t)I(t+\tau)} \tag{2.5-5}$$

This introduces a considerable simplification and it appears that the simplest method of computing $\bar{w}(f)$ for an $I(t)$ of this sort is first to compute $\bar{\psi}(\tau)$, and then use the inversion formula (2.5-1).

2.6 First Example—The Shot Effect

We first compute the average on the right of (2.5-5). By using the method of averaging employed many times in part I, we have

$$\overline{I(t)I(t+\tau)} = \sum_{K=0}^{\infty} p(K)\ \overline{I_K(t)I_K(t+\tau)} \tag{2.6-1}$$

where $p(K)$ is the probability of exactly K electrons arriving in the interval $(0, T)$,

$$p(K) = \frac{(\nu T)^K}{K!} e^{-\nu T} \tag{1.1-3}$$

and

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.3-1}$$

Multiplying $I_K(t)$ and $I_K(t + \tau)$ together and averaging $t_1, t_2, \cdots t_K$ over their ranges gives

$$\overline{I_K(t) I_K(t + \tau)} = \sum_{k=1}^{K} \sum_{m=1}^{K} \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T} F(t - t_k) F(t + \tau - t_m)$$

This is similar to the expression for $\overline{I_K^2(t)}$ which was used in section 1.3 to prove Campbell's theorem and may be treated in much the same way. Thus, if t and $t + \tau$ lie between Δ and $T - \Delta$, the expression above becomes

$$\frac{K}{T} \int_{-\infty}^{+\infty} F(t) F(t + \tau)\, dt + \frac{K(K-1)}{T^2} \left[\int_{-\infty}^{+\infty} F(t)\, dt \right]^2$$

When this is placed in (2.6-1) and the summation performed we obtain an expression independent of T. Consequently we may use (2.5-5) and get

$$\bar{\psi}(\tau) = \nu \int_{-\infty}^{+\infty} F(t) F(t + \tau)\, dt + \overline{I(t)}^2 \tag{2.6-2}$$

where we have used the expression for the average current

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.3-4}$$

In order to compute $\bar{w}(f)$ from $\bar{\psi}(\tau)$ it is convenient to make use of the fact that $\psi(\tau)$ is always an even function of τ and hence (2.5-1) may also be written as

$$\bar{w}(f) = 2 \int_{-\infty}^{+\infty} \bar{\psi}(\tau) \cos 2\pi f\tau\, d\tau \tag{2.6-3}$$

Then

$$\bar{w}(f) = 2\nu \int_{-\infty}^{+\infty} dt\, F(t) \int_{-\infty}^{+\infty} d\tau\, F(t + \tau) \cos 2\pi f\tau$$
$$+ 2 \int_{-\infty}^{+\infty} \overline{I(t)}^2 \cos 2\pi f\tau\, d\tau$$

$$= 2\nu \text{ Real Part of } \int_{-\infty}^{+\infty} dt\, F(t)e^{-2\pi i f t} \int_{-\infty}^{+\infty} dt'\, F(t')e^{2\pi i f t'}$$

$$+ 2\overline{I(t)}^2 \int_{-\infty}^{+\infty} e^{i2\pi f\tau}\, d\tau$$

$$= 2\nu \,|\, s(f) \,|^2 + 2\overline{I(t)}^2 \delta(f) \tag{2.6-4}$$

In going from the first equation to the second we have written $t' = t + \tau$ and have considered $\cos 2\pi f\tau$ to be the real part of the corresponding exponential. In going from the second equation to the third we have set

$$s(f) = \int_{-\infty}^{+\infty} F(t)e^{-2\pi i f t}\, dt \tag{2.6-5}$$

and have used

$$\int_{-\infty}^{+\infty} e^{i2\pi f t}\, dt = \delta(f) \tag{2.2-9}$$

The term in $\bar{w}(f)$ involving $\delta(f)$ represents the average power which would be dissipated by the d.c. component of $I(t)$ in flowing through one ohm. It is in agreement with the concept that the average power in the band $0 \le f < \epsilon$, $\epsilon > 0$ but very small, is

$$\int_0^{\epsilon} \bar{w}(f)\, dt = 2\bar{I}(t)^2 \int_0^{\epsilon} \delta(f)\, df$$
$$= \bar{I}(t)^2 \tag{2.6-6}$$

The expression (2.6-4) for $\bar{w}(f)$ may also be obtained from the definition (2.5-3) for $\bar{w}(f)$plus the additional term due to the d.c. component obtained by averaging the expressions (2.2-11). We leave this as an exercise for the reader. He will find it interesting to study the steps in Carson's[15] paper leading up to equation (8). Carson's $R(\omega)$ is related to our $\bar{w}(f)$ by

$$\bar{w}(f) = 2\pi R(\omega)$$

and his $f(i\omega)$ is equal to our $s(f)$.

Integrating both sides of (2.6-4) with respect to f from 0 to ∞ and using

$$\bar{I^2} = \int_0^{\infty} \bar{w}(f)\, df$$

gives the result

$$\bar{I^2} - \bar{I}^2 = 2\nu \int_0^{+\infty} |\, s(f)\,|^2\, df \tag{2.6-7}$$

[15] Loc. cit.

This may be obtained immediately from Campbell's theorem by applying Parseval's theorem.

As an example of the use of these formulas we derive the power spectrum of the voltage across a resistance R when a current consisting of a great number of very short pulses per second flows through R. Let $F(t - t_k)$ be the voltage produced by the pulse occurring at time t_k. Then

$$F(t) = R\varphi(t)$$

where $\varphi(t)$ is the current in the pulse. We confine our interest to relatively low frequencies such that we may make the approximation

$$s(f) = \int_{-\infty}^{+\infty} R\varphi(t)e^{-2\pi ift}\,dt$$

$$\approx R\int_{-\infty}^{+\infty} \varphi(t)\,dt = Rq$$

where q is the charge carried through the resistance by one pulse. From (2.6-4) it follows that for these low frequencies the continuous portion of the power spectrum for the voltage is constant and equal to

$$\bar{w}(f) = 2\nu R^2q^2 = 2\bar{I}R^2q \tag{2.6-8}$$

where $\bar{I} = \nu q$ is the average current flowing through R. This result is often used in connection with the shot effect in diodes.

In the study of the shot effect it was assumed that the probability of an event (electron arriving at the anode) happening in dt was νdt where ν is the expected number of events per second. This probability is independent of the time t. Sometimes we wish to introduce dependency on time.[18] As an example, consider a long interval extending from 0 to T. Let the probability of an event happening in $t, t + dt$ be $\bar{K}p(t)dt$ where $\bar{K}$ is the average number of events during T and $p(t)$ is a given function of t such that

$$\int_0^T p(t)\,dt = 1$$

For the shot effect $p(t) = 1/T$.

What is the probability that exactly K events happen in T? As in the case of the shot effect, section 1.1, we may divide $(0, T)$ into N intervals each of length Δt so that $N\Delta t = T$. The probability of no event happening in the first Δt is

$$1 - \bar{K}p\left(\frac{\Delta t}{2}\right)\Delta t$$

[18] A careful discussion of this subject is given by Hurwitz and Kac in "Statistical Analysis of Certain Types of Random Functions." I understand that this paper will soon appear in the Annals of Math. Statistics.

The product of N such probabilities is, as $N \to \infty$, $\Delta t \to 0$;

$$\exp\left[-\bar{K}\int_0^T p(t)\,dt\right] = e^{-\bar{K}}$$

This is the probability that exactly 0 events happen in T. In the same way we are led to the expression

$$\frac{\bar{K}^K}{K!} e^{-\bar{K}} \tag{2.6-9}$$

for the probability that exactly K events happen in T.

When we consider many intervals $(0, T)$ we obtain many values of K and also many values of I measured t seconds from the beginning of each interval. These values of I define the distribution of I at time t. By proceeding as in section 1.4 we find that the probability density of I is

$$P(I, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \exp\left[-iuI + \bar{K}\int_0^T p(x)(e^{iuF(t-x)} - 1)\,dx\right]$$

The corresponding average and variance is

$$\bar{I} = \bar{K}\int_0^T p(x)F(t-x)\,dx$$

$$\overline{(I - \bar{I})^2} = \bar{K}\int_0^T p(x)F^2(t-x)\,dx \tag{2.6-10}$$

If $S(f)$ is given by (2.1-2) and $s(f)$ by (2.6-5) (assuming the duration of $F(t)$ short in comparison with T) the average value of $|S(f)|^2$ may be obtained by putting (1.3-1) in (2.1-2) to get

$$S_K(f) = s(f)\sum_1^K e^{-2\pi i f t_k}$$

Expressing $S_K(f)\, S_K^*(f)$, where the star denotes conjugate complex, as a double sum and averaging over the t_k's, using $p(t)$, and then averaging over the K's gives

$$\overline{|S(f)|^2} = \bar{K}\,|s(f)|^2\left[1 + \bar{K}\left|\int_0^T p(x)e^{-2\pi i f x}\,dx\right|^2\right] \tag{2.6-11}$$

This may be used to compute the power spectrum from (2.5-3) provided $p(x)$ is not periodic. If $p(x)$ is periodic then the method of section 2.2 should be used at the harmonic frequencies. If the fluctuations of $p(t)$ are slow in comparison with the fluctuations of $F(t)$ the second term within the brackets of (2.6-11) may generally be neglected since there are no values of

f which make both it and $s(f)$ large at the same time. On the other hand, if both $p(t)$ and $I(t)$ fluctuate at about the same rate this term must be considered.

2.7 Second Example. Random Telegraph Signal[16]

Let $I(t)$ be equal to either a or $-a$ so that it is of the form of a flat top wave. Let the intervals between changes of sign, i.e. the lengths of the tops and bottoms, be distributed exponentially. We are led to this distribution by assuming that, if on the average there are μ changes of sign per second, the probability of a change of sign in $t, t + dt$ is μdt and is independent of what happens outside the interval $t, t + dt$. From the same sort of reasoning as employed in section 1.1 for the shot effect we see that the probability of obtaining exactly K changes of sign in the interval $(0, T)$ is

$$p(K) = \frac{(\mu T)^K}{K!} e^{-\mu T} \tag{2.7–1}$$

We consider the average value of the product $I(t)I(t + \tau)$. This product is a^2 if the two I's are of the same sign and is $-a^2$ if they are of opposite sign. In the first case there are an even number, including zero, of changes of sign in the interval $(t, t + \tau)$, and in the second case there are an odd number of changes of sign. Thus

$$\begin{aligned} &\text{Average value of } I(t)I(t + \tau) \\ &= a^2 \times \text{probability of an even number of changes of sign in } t, t + \tau \\ &\quad - a^2 \times \text{probability of an odd number of changes of sign in } t, t + \tau \end{aligned} \tag{2.7–2}$$

The length of the interval under consideration is $|t + \tau - t| = |\tau|$ seconds. Since, by assumption, the probability of a change of sign in an elementary interval of length Δt is independent of what happens outside that interval, it follows that the same is true of any interval irrespective of when it starts. Hence the probabilities in (2.7–2) are independent of t and may be obtained from (2.7–1) by setting $T = |\tau|$. Then (2.7–2) becomes, assuming $\tau > 0$ for the moment,

$$\begin{aligned} \overline{I(t)I(t + \tau)} &= a^2[p(0) + p(2) + p(4) + \cdots] \\ &\qquad - a^2[p(1) + p(3) + p(5) + \cdots] \\ &= a^2 e^{-\mu\tau}\left[1 - \frac{\mu\tau}{1!} + \frac{(\mu\tau)^2}{2!} - \cdots\right] \\ &= a^2 e^{-2\mu\tau} \end{aligned} \tag{2.7–3}$$

[16] Kenrick, cited in Section 2.2.

From (2.5-5), this gives the correlation function for $I(t)$

$$\bar{\psi}(\tau) = a^2 e^{-2\mu|\tau|} \tag{2.7-4}$$

The corresponding power spectrum is, from (2.5-1),

$$\begin{aligned} \bar{w}(f) &= 4a^2 \int_0^\infty e^{-2\mu\tau} \cos 2\pi f\tau \, d\tau \\ &= \frac{2a^2 \mu}{\pi^2 f^2 + \mu^2} \end{aligned} \tag{2.7-5}$$

Correlation functions and power spectra of this type occur quite frequently. In particular, they are of use in the study of turbulence in hydrodynamics. We may also obtain them from our shot effect expressions if we disregard the d.c. component. All we have to do is to assume that the effect $F(t)$ of an electron arriving at the anode at time $t = 0$ is zero for $t < 0$, and that $F(t)$ decays exponentially with time after jumping to its maximum value at $t = 0$. This may be verified by substituting the value

$$F(t) = 2a \sqrt{\frac{\mu}{\nu}}\, e^{-2\mu t}, \qquad t > 0 \tag{2.7-6}$$

for $F(t)$ in the expressions (2.6-2) and (2.6-4) (after using 2.6-5) for the correlation function and energy spectrum of the shot effect.

The power spectrum of the current flowing through an inductance and a resistance in series in response to a very wide band thermal noise voltage is also of the form (2.7-5).

Incidentally, this gives us an example of two quite different $I(t)$'s, one a flat top wave and the other a shot effect current, which have the same correlation functions and power spectra, aside from the d.c. component.

There is another type of random telegraph signal which is interesting to analyze. The time scale is divided into intervals of equal length h. In an interval selected at random the value of $I(t)$ is independent of the values in the other intervals, and is equally likely to be $+a$ or $-a$. We could construct such a wave by flipping a penny. If heads turned up we would set $I(t) = a$ in $0 < t < h$. If tails were obtained we would set $I(t) = -a$ in this interval. Flipping again would give either $+a$ or $-a$ for the second interval $h < t < 2h$, and so on. This gives us one wave. A great many waves may be constructed in this way and we denote averages over these waves, with t held constant, by bars.

We ask for the average value of $I(t)I(t + \tau)$, assuming $\tau > 0$. First we note that if $\tau > h$ the currents correspond to different intervals for all values of t. Since the values in these intervals are independent we have

$$\overline{I(t)I(t+\tau)} = \overline{I(t)}\ \overline{I(t+\tau)} = 0$$

for all values of t when $\tau > h$.

To obtain the average when $\tau < h$ we consider t to lie in the first interval $0 < t < h$. Since all the intervals are the same from a statistical point of view we lose no generality in doing this. If $t + \tau < h$, i.e., $t < h - \tau$, both currents lie in the first interval and

$$\overline{I(t)I(t+\tau)} = a^2$$

If $t > h - \tau$ the current $I(t + \tau)$ corresponds to the second interval and hence the average value is zero.

We now return to (2.5-4). The integral there extends from 0 to T. When $\tau > h$, the integrand is zero and hence

$$\bar{\psi}(\tau) = 0, \qquad \tau > h \tag{2.7-7}$$

When $\tau < h$, our investigation of the interval $0 < t < h$ enables us to write down the portion of the integral extending from 0 to h:

$$\int_0^h I(t)I(t+\tau)\,dt = \int_0^{h-\tau} a^2\,dt + \int_{h-\tau}^h 0\,dt$$
$$= a^2(h - \tau)$$

Over the interval of integration $(0, T)$ we have T/h such intervals each contributing the same amount. Hence, from (2.5-4),

$$\begin{aligned}\bar{\psi}(\tau) &= \operatorname*{Limit}_{T\to\infty} \frac{a^2}{T}\cdot\frac{T}{h}(h - \tau) \\ &= a^2\left(1 - \frac{\tau}{h}\right), \qquad 0 \le \tau < h\end{aligned} \tag{2.7-8}$$

The power spectrum of this type of telegraph wave is thus

$$\begin{aligned}\bar{w}(f) &= 4a^2\int_0^h \left(1 - \frac{\tau}{h}\right)\cos 2\pi f\tau\,d\tau \\ &= 2h\left(\frac{a\sin \pi fh}{\pi fh}\right)^2\end{aligned} \tag{2.7-9}$$

This is seen to have the same general behavior as $\bar{w}(f)$ for the first type of telegraph signal given by (2.7-5), when we relate the average number, μ, of changes of sign per second to the interval length h by $\mu h = 1$.

In section 1.8 the Fourier series representation of the shot effect current was discussed. This suggests the representation*

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \tag{2.8-1}$$

where

$$\omega_n = 2\pi f_n, \qquad f_n = n\Delta f \tag{2.8-2}$$

a_n and b_n are taken to be independent random variables which are distributed normally about zero with the standard deviation $\sqrt{w(f_n)\Delta f}$. $w(f)$ is the power spectrum of the noise current, i.e., $w(f)\,df$ is the average power which would be dissipated by those components of $I(t)$ which lie in the frequency range $f, f + df$ if they were to flow through a resistance of one ohm.

The expression for the standard deviation of a_n and b_n is obtained when we notice that Δf is the width of the frequency band associated with the nth component. Hence $w(f_n)\Delta f$ is the average energy which would be dissipated if the current

$$a_n \cos \omega_n t + b_n \sin \omega_n t$$

were to flow through a resistance of one ohm, this average being taken over all possible values of a_n and b_n. Thus

$$w(f_n)\Delta f = \overline{a_n^2 \cos^2 \omega_n t + 2a_n b_n \cos \omega_n t \sin \omega_n t + b_n^2 \sin^2 \omega_n t} = \overline{a_n^2} = \overline{b_n^2} \tag{2.8-3}$$

The last two steps follow from the independence of a_n and b_n and the identity of their distributions. It will be observed that $w(f)$, as used with the representation (2.8-1), is the same sort of average as was denoted in section 2.5 by $\bar{w}(f)$. However, $w(f)$ is often given to us in order to specify the spectrum of a given noise.

For example, suppose we are interested in the output of a certain filter when a source of thermal noise is applied to the input. Let $A(f)$ be the absolute value of the ratio of the output current to the input current when a steady sinusoidal voltage of frequency f is applied to the input. Then

$$w(f) = cA^2(f) \tag{2.8-4}$$

* As mentioned in section 1.7 this sort of representation was used by Einstein and Hopf for radiation. Shottky (1918) used (2.8-1), apparently without explicitly taking the coefficients to be normally distributed. Nyquist (1932) derived the normal distribution from the shot effect.

If W is the average power dissipated in one ohm by $I(t)$,

$$W = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I^2(t)\,dt = \int_0^\infty w(f)\,df$$
$$= c\int_0^\infty A^2(f)\,df \qquad (2.8\text{-}5)$$

which is an equation to determine c when W and $A(f)$ are known.

In using the representation (2.8-1) to investigate the statistical properties of $I(t)$ we first find the corresponding statistical properties of the summation on the right when the a's and b's are regarded as random variables distributed as mentioned above and t is regarded as fixed. In general, the time t disappears in this procedure just as it did in (2.8-3). We then let $N \to \infty$ and $\Delta f \to 0$ so that the summations may be replaced by integrations. Finally, the frequency range is extended to cover all frequencies from 0 to ∞.

The usual way of looking at the representation (2.8-1) is to suppose that we have an oscillogram of $I(t)$ extending from $t = 0$ to $t = \infty$. This oscillogram may be cut up into strips of length T. A Fourier analysis of $I(t)$ for each strip will give a set of coefficients. These coefficients will vary from strip to strip. Our representation ($T\Delta f = 1$) assumes that this variation is governed by a normal distribution. Our process for finding statistical properties by regarding the a's and b's as random variables while t is kept fixed corresponds to examining the noise current at a great many instants. Corresponding to each strip there is an instant, and this instant occurs at t (this is the t in (2.8-1)) seconds from the beginning of the strip. This is somewhat like examining the noise current at a great number of instants selected at random.

Although (2.8-1) is the representation which is suggested by the shot effect and similar phenomena, it is not the only representation, nor is it always the most convenient. Another representation which leads to the same results when the limits are taken is[10]

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n) \qquad (2.8\text{-}6)$$

where $\varphi_1, \varphi_2, \cdots \varphi_N$ are angles distributed at random over the range $(0, 2\pi)$ and

$$c_n = [2w(f_n)\Delta f]^{1/2}, \qquad \omega_n = 2\pi f_n, \qquad f_n = n\Delta f \qquad (2.8\text{-}7)$$

[10] This representation has often been used by W. R. Bennett in unpublished memoranda written in the 1930's.

In this representation $I(t)$ is regarded as the sum of a number of sinusoidal components with fixed amplitudes but random phase angles.

That the two different representations (2.8-1) and (2.8-6) of $I(t)$ lead to the same statistical properties is a consequence of the fact that they are always used in such a way that the "central limit theorem*" may be used in both cases.

This theorem states that under certain general conditions, the distribution of the sum of N random vectors approaches a normal law (it may be normal in several dimensions**) as $N \to \infty$. In fact from this theorem it appears that a representation such as

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad (2.8\text{-}6)$$

where a_n and b_n are independent random variables which take only the values $\pm [w(f_n)\Delta f]^{1/2}$, the probability of each value being $\frac{1}{2}$, will lead in the limit to the same statistical properties of $I(t)$ as do (2.8-1) and (2.8-6).

2.9 The Normal Distribution in Several Variables[20]

Consider a random vector r in K dimensions. The distribution of this vector may be specified by stating the distribution of the K components, $x_1, x_2, \cdots x_K$, of r. r is said to be normally distributed when the probability density function of the x's is of the form

$$(2\pi)^{-K/2} | M |^{-1/2} \exp [-\tfrac{1}{2} x' M^{-1} x] \qquad (2.9\text{-}1)$$

where the exponent is a quadratic form in the x's. The square matrix M is composed of the second moments of the x's.

$$M = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1K} \\ \cdot & & & \cdot \\ \mu_{1K} & & \cdots & \mu_{KK} \end{bmatrix} \qquad (2.9\text{-}2)$$

where the second moments are defined by

$$\mu_{11} = \overline{x_1^2}, \qquad \mu_{12} = \overline{x_1 x_2}, \qquad \text{etc.} \qquad (2.9\text{-}3)$$

$| M |$ represents the determinant of M and x' is the row matrix

$$x' = [x_1, x_2, \cdots x_K] \qquad (2.9\text{-}4)$$

x is the column matrix obtained by transposing x'.

* See section 2.10.
** See section 2.9.
[20] H. Cramér, "Random Variables and Probability Distributions." Chap. X., Cambridge Tract No. 36 (1937).

The exponent in the expression (2.9-1) for the probability density may be written out by using

$$x'M^{-1}x = \sum_{r=1}^{K}\sum_{s=1}^{K}\frac{M_{rs}}{|M|}x_r x_s \qquad (2.9\text{-}5)$$

where M_{rs} is the cofactor of μ_{rs} in M.

Sometimes there are linear relations between the x's so that the random vector $\mathbf{r}$ is restricted to a space of less than K dimensions. In this case the appropriate form for the density function may be obtained by considering a sequence of K-dimensional distributions which approach the one being investigated.

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are two normally distributed random vectors their sum $\mathbf{r}_1 + \mathbf{r}_2$ is also normally distributed. It follows that the sum of any number of normally distributed random vectors is normally distributed.

The characteristic function of the normal distribution is

$$\text{ave. } e^{iz_1x_1+iz_2x_2+\cdots+iz_Kx_K} = \exp\left[-\frac{1}{2}\sum_{r=1}^{K}\sum_{s=1}^{K}\mu_{rs}z_r z_s\right] \qquad (2.9\text{-}6)$$

2.10 Central Limit Theorem

The central limit theorem in probability states that the distribution of the sum of N independent random vectors $\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_N$ approaches a normal law as $N \to \infty$ when the distributions of $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N$ satisfy certain general conditions.[7]

As an example we take the case in which $\mathbf{r}_1, \mathbf{r}_2, \cdots$ are two-dimensional vectors[21], the components of $\mathbf{r}_n$ being x_n and y_n. Without loss of generality we assume that

$$\bar{x}_n = 0, \qquad \bar{y}_n = 0.$$

The components of the resultant vector are

$$\begin{aligned} X &= x_1 + x_2 + \cdots + x_N \\ Y &= y_1 + y_2 + \cdots + y_N \end{aligned} \qquad (2.10\text{-}1)$$

and, since $\mathbf{r}_1, \mathbf{r}_2, \cdots$ are independent vectors, the second moments of the resultant are

$$\begin{aligned} \mu_{11} &= \overline{X^2} = \overline{x_1^2} + \overline{x_2^2} + \cdots + \overline{x_N^2} \\ \mu_{22} &= \overline{Y^2} = \overline{y_1^2} + \overline{y_2^2} + \cdots + \overline{y_N^2} \\ \mu_{12} &= \overline{XY} = \overline{x_1y_1} + \overline{x_2y_2} + \cdots + \overline{x_Ny_N} \end{aligned} \qquad (2.10\text{-}2)$$

[7] Incidentally, von Laue (see references in section 1.7) used this theorem in discussing the normal distribution of the coefficients in a Fourier series used to represent black-body radiation. He ascribed it to Markoff.

[21] This case is discussed by J. V. Uspensky, "Introduction to Mathematical Probability", McGraw-Hill (1937) Chap. XV.

Apparently there are several types of conditions which are sufficient to ensure that the distribution of the resultant approaches a normal law. One sufficient condition is that[21]

$$\mu_{11}^{-3/2} \sum_{n=1}^{N} | x_n |^3 \to 0$$
$$\mu_{22}^{-3/2} \sum_{n=1}^{N} | y_n |^3 \to 0 \qquad (2.10\text{-}3)$$

The central limit theorem tells us that the distribution of the random vector (X, Y) approaches a normal law as $N \to \infty$. The second moments of this distribution are given by (2.10-2). When we know the second moments of a normal distribution we may write down the probability density function at once. Thus from section 2.9

$$M = \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{bmatrix}, \qquad M^{-1} = | M |^{-1} \begin{bmatrix} \mu_{22} & -\mu_{12} \\ -\mu_{12} & \mu_{11} \end{bmatrix}$$

$$| M | = \mu_{11}\mu_{22} - \mu_{12}^2$$

$$x' = [X, Y]$$

$$x'M^{-1}x = | M |^{-1}(\mu_{22}X^2 - 2\mu_{12}XY + \mu_{11}Y^2)$$

The probability density is therefore

$$\frac{(\mu_{11}\mu_{22} - \mu_{12}^2)^{-1/2}}{2\pi} \cdot \exp\left[\frac{-\mu_{22}X^2 - \mu_{11}Y^2 + 2\mu_{12}XY}{2(\mu_{11}\mu_{22} - \mu_{12}^2)}\right] \qquad (2.10\text{-}3)$$

Incidentally, the second moments are related to the standard deviations σ_1, σ_2 of X, Y and to the correlation coefficient r of X and Y by

$$\mu_{11} = \sigma_1^2, \qquad \mu_{22} = \sigma_2^2, \qquad \mu_{12} = r\sigma_1\sigma_2 \qquad (2.10\text{-}4)$$

and the probability density takes the standard form

$$\frac{(1 - r^2)^{-1/2}}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2(1 - r^2)}\left(\frac{X^2}{\sigma_1^2} - 2r\frac{XY}{\sigma_1\sigma_2} + \frac{Y^2}{\sigma_2^2}\right)\right] \qquad (2.10\text{-}5)$$

[21] This is used by Uspensky, loc. cit. Another condition analogous to the Lindeberg condition is given by Cramer,[26] loc. cit.

PART III

STATISTICAL PROPERTIES OF RANDOM NOISE CURRENTS

3.0 Introduction

In this section we use the representations of the noise currents given in section 2.8 to derive some statistical properties of $I(t)$. The first six sections are concerned with the probability distribution of $I(t)$ and of its zeros and maxima. Sections 3.7 and 3.8 are concerned with the statistical properties of the envelope of $I(t)$. Fluctuations of integrals involving $I^2(t)$ are discussed in section 3.9. The probability distribution of a sine wave plus a noise current is given in 3.10 and in 3.11 an alternative method of deriving the results of Part III is mentioned. Prof. Uhlenbeck has pointed out that much of the material in this Part is closely connected with the theory of Markoff processes. Also S. Chandrasekhar has written a review of a class of physical problems which is related, in a general way, to the present subject.[22]

3.1 The Distribution of the Noise Current[23]

In section 1.4 it has been shown that the distribution of a shot effect current approaches a normal law as the expected number of events per second, ν, increases without limit.

In line with the spirit of this Part, Part III, we shall use the representation

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad (2.8\text{-}1)$$

to show that $I(t)$ is distributed according to a normal law. This is obtained at once when the procedure outlined in section 2.8 is followed. Since a_n and b_n are distributed normally, so are $a_n \cos \omega_n t$ and $b_n \sin \omega_n t$ when t is regarded as fixed. $I(t)$ is thus the sum of $2N$ independent normal variates and consequently is itself distributed normally.

[22] Stochastic Problems in Physics and Astronomy, *Rev. of Mod. Phys.*, Vol. 15, pp. 1–89 (1943).

[23] An interesting discussion of this subject by V. D. Landon and K. A. Norton is given in the *I.R.E. Proc.*, 30 (Sept. 1942) pp. 425–429.

The average value of $I(t)$ as given by (2.8-1) is zero since $\bar{a}_n = \bar{b}_n = 0$:

$$\bar{I}(t) = 0 \tag{3.1-1}$$

The mean square value of $I(t)$ is

$$\begin{aligned}\overline{I^2(t)} &= \sum_{n=1}^{N} (\overline{a_n^2} \cos^2 \omega_n t + \overline{b_n^2} \sin^2 \omega_n t) \\ &= \sum_{n=1}^{N} w(f_n)\Delta f \qquad (3.1\text{-}2) \\ &\to \int_0^\infty w(f)\, df = \psi(0) = \psi_0\end{aligned}$$

In writing down (3.1-2) we have made use of the fact that all the a's and b's are independent and consequently the average of any cross product is zero. We have also made use of

$$\overline{a_n^2} = \overline{b_n^2} = w(f_n)\Delta f, \qquad f_n = n\Delta f, \qquad \omega_n = 2\pi f_n$$

which were given in 2.8. $\psi(\tau)$ is the correlation function of $I(t)$ and is related to $w(f)$ by

$$\psi_\tau = \psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1-6}$$

as is explained in section 2.1. In this part we shall write the argument of $\psi(\tau)$ as a subscript in order to save space.

Since we know that $I(t)$ is normal and since we also know that its average is zero and its mean square value is ψ_0, we may write down its probability density function at once. Thus, the probability of $I(t)$ being in the range $I, I + dI$ is

$$\frac{dI}{\sqrt{2\pi\psi_0}}\, e^{-I^2/2\psi_0} \tag{3.1-3}$$

This is the probability of finding the current between I and $I + dI$ at a time selected at random. Another way of saying the same thing is to state that (3.1-3) is the fraction of time the current spends in the range $I, I + dI$.

In many cases it is more convenient to use the representation (2.8-6)

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n), \qquad \overline{c_n^2} = 2w(f_n)\Delta f \tag{2.8-6}$$

in which $\varphi_1, \cdots \varphi_n$ are independent random phase angles. In order to deduce the normal distribution from this representation we first observe that (2.8–6) expresses $I(t)$ as the sum of a large number of independent random variables

$$I(t) = x_1 + x_2 + \cdots + x_N$$
$$x_n = c_n \cos(\omega_n t - \varphi_n)$$

and hence that as $N \to \infty$ $I(t)$ becomes distributed according to a normal law. In order to make the limiting process definite we first choose N and Δf such that $N\Delta f = F$ where

$$\int_F^\infty w(f)\,df < \epsilon \int_0^\infty w(f)\,df$$

where ϵ is some arbitrarily chosen small positive quantity. We now let $N \to \infty$ and $\Delta f \to 0$ in such a way that $N\Delta f$ remains equal to F. Then

$$\begin{aligned} A = \overline{x_1^2} + \overline{x_2^2} + \cdots + \overline{x_N^2} &= \sum_1^N 2w(f_n)\Delta f\, \overline{\cos^2(\omega_n t - \varphi_n)} \\ &= \sum_1^N w(f_n)\Delta f \to \int_0^F w(f)\,df \end{aligned} \tag{3.1–4}$$

$$\begin{aligned} B = \overline{|x_1|^3} + \cdots + \overline{|x_N|^3} &= \sum_1^N (2w(f_n)\Delta f)^{3/2}\,\overline{|\cos(\omega_n t - \varphi_n)|^3} \\ &< 4(\Delta f)^{1/2}\int_0^F [w(f)]^{3/2}\,df \end{aligned}$$

where the bars denote averages with respect to the φ's, t being held constant. If we assume that the integrals are proper, the ratio $BA^{-3/2} \to 0$ as $N \to \infty$, and consequently the central limit theorem* may be used if $w(f) = 0$ for $f > F$. Since we may make F as large as we please by choosing ϵ small enough, we may cover as large a frequency range as we wish. For this reason we write ∞ in place of F.

Now that the central limit theorem has told us that the distribution of $I(t)$, as given by (2.8–6), approaches a normal law, there remains only the problem of finding the average and the standard deviation:

$$\overline{I(t)} = \sum_1^N c_n \overline{\cos(\omega_n t - \varphi_n)} = 0$$

$$\begin{aligned} \overline{I^2(t)} &= \sum_1^N c_n^2\, \overline{\cos^2(\omega_n t - \varphi_n)} \\ &\to \int_0^\infty w(f)\,df = \psi_0 \end{aligned} \tag{3.1–5}$$

* Section 2.10.

This gives the probability density (3.1-3). Hence the two representations lead to the same result in this case. Evidently, they will continue to lead to identical results as long as the central limit theorem may be used. In the future use of the representation (2.8-6) we shall merely assume that the central limit theorem may be applied to show that a normal distribution is approached. We shall omit the work corresponding to equations (3.1-4).

The characteristic function for the distribution of $I(t)$ is

$$\text{ave. } e^{iuI(t)} = \exp - \frac{\psi_0}{2} u^2 \tag{3.1-6}$$

3.2 The Distribution of $I(t)$ and $I(t + \tau)$

We require the two dimensional distribution in which the first variable is the noise current $I(t)$ and the second variable is its value $I(t + \tau)$ at some later time τ. It turns out that this distribution is normal[24], as we might expect from the analogy with section 3.1. The second moments of this distribution are

$$\begin{aligned}
\mu_{11} &= \overline{I^2(t)} = \psi_0 = \int_0^\infty w(f)\, df \\
\mu_{22} &= \psi_0 \\
\mu_{12} &= \overline{I(t)I(t + \tau)} \\
&= \psi_\tau
\end{aligned} \tag{3.2-1}$$

The expression for μ_{12} is in line with our definition (2.1-4) for the correlation function:

$$\psi_\tau \equiv \psi(\tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \tag{2.1-4}$$

In order to get the distribution from the representation (2.8-6) we write

$$I_1 = I(t) = \sum_1^N c_n \cos(\omega_n t - \varphi_n)$$

$$I_2 = I(t + \tau) = \sum_1^N c_n \cos(\omega_n t - \varphi_n + \omega_n \tau)$$

[24] It seems that the first person to obtain this distribution in connection with noise was H. Thiede, *Elec. Nachr. Tek.* 13 (1936), 84-95.

From the central limit theorem for two dimensions it follows that I_1 and I_2 are distributed normally. As in (3.1)

$$\mu_{11} = \overline{I_1^2} = \sum_1^N c_n^2 \cdot \tfrac{1}{2} \rightarrow \int_0^\infty w(f)\, df = \psi_0$$

$$\mu_{22} = \overline{I_2^2} = \overline{I_1^2} = \psi_0 \tag{3.1-2}$$

$$\mu_{12} = \overline{I_1 I_2} = \sum_1^N c_n^2 \text{ ave. } \{\cos(\omega_n t - \varphi_n) \cos(\omega_n t - \varphi_n + \omega_n \tau)\}$$

Now the quantity within the parenthesis is

$$\cos^2(\omega_n t - \varphi_n) \cos \omega_n \tau - \cos(\omega_n t - \varphi_n) \sin(\omega_n t - \varphi_n) \sin \omega_n \tau$$

and when we take the average with respect to φ_n the second term drops out, giving

$$\mu_{12} = \sum_1^N c_n^2 \cdot \tfrac{1}{2} \cos \omega_n \tau \rightarrow \int_0^\infty w(f) \cos 2\pi f \tau\, df = \psi_\tau \tag{3.2-3}$$

where we have used $\omega_n = 2\pi f_n$ and the relation (2.1-6) between $w(f)$ and $\psi(\tau)$.

The probability density function for I_1 and I_2 may be stated. From the discussion of the normal law in 2.9 it is

$$\frac{[\psi_0^2 - \psi_\tau^2]^{-1/2}}{2\pi} \exp\left[\frac{-\psi_0 I_1^2 - \psi_0 I_2^2 + 2\psi_\tau I_1 I_2}{2(\psi_0^2 - \psi_\tau^2)}\right] \tag{3.2-4}$$

For a band pass filter whose range extends from f_a to f_b we have

$$\begin{aligned} \psi_\tau &= \int_{f_a}^{f_b} w_0 \cos 2\pi f \tau\, df \\ &= w_0 \frac{\sin \omega_b \tau - \sin \omega_a \tau}{2\pi\tau} \\ &= \frac{w_0}{\pi\tau} \sin \pi\tau(f_b - f_a) \cos \pi\tau(f_b + f_a) \\ \psi_0 &= w_0(f_b - f_a) \end{aligned} \tag{3.2-5}$$

where w_0 is the constant value of $w(f)$ in the pass band and

$$\begin{aligned} \omega_b &= 2\pi f_b \\ \omega_a &= 2\pi f_a \end{aligned} \tag{3.2-6}$$

According to our formula (3.2-4), I_1 and I_2 are independent when ψ_τ is zero. For the τ's which make ψ_τ zero, a knowledge of I_1 does not add to our knowledge of I_2. For example, suppose we have a narrow filter. Then

$$\psi_\tau = 0 \text{ when } \tau = [2(f_b + f_a)]^{-1}$$

$$\psi_\tau \text{ is nearly } -\psi_0 \text{ when } \tau = [f_b + f_a]^{-1}$$

For the first value of τ, all we know is that I_2 is distributed about zero with $\overline{I_2^2} = \psi_0$. For the second value of τ I_2 is likely to be near $-I_1$. This is in line with the idea that the noise current through a narrow filter behaves like a sine wave of frequency $\frac{1}{2}(f_b + f_a)$ (and, incidentally, whose amplitude fluctuates with an irregular frequency of the order of $\frac{1}{2}(f_b - f_a)$). The first value of τ corresponds to a quarter-period of such a wave and the second value to a half-period. By drawing a sine wave and looking at points separated by quarter and half periods, the reader will see how the ideas agree.

The characteristic function for the distribution of I_1 and I_2 is

$$\text{ave. } e^{iuI_1 + ivI_2} = \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \tag{3.2-7}$$

The three dimensional distribution in which

$$I_1 = I(t)$$
$$I_2 = I(t + \tau_1)$$
$$I_3 = I(t + \tau_1 + \tau_2)$$

where τ_1 and τ_2 are given and t is chosen at random is, as we might expect, normal in three dimensions. The moments, from which the distribution may be obtained by the method of Section 2.9, are

$$\mu_{11} = \mu_{22} = \mu_{33} = \psi_0$$
$$\mu_{12} = \psi_{\tau_1}$$
$$\mu_{23} = \psi_{\tau_2}$$
$$\mu_{13} = \psi(\tau_1 + \tau_2) = \psi_{\tau_1+\tau_2}$$

The characteristic function for I_1, I_2, I_3 is

$$\text{ave. } e^{iz_1I_1 + iz_2I_2 + iz_3I_3}$$

$$= \exp\left[-\frac{\psi_0}{2}(z_1^2 + z_2^2 + z_3^2) - \mu_{12}z_1z_2 - \mu_{23}z_2z_3 - \mu_{13}z_1z_3\right] \tag{3.2-8}$$

3.3 Expected Number of Zeros per Second

We shall use the following result. Let y be given by

$$y = F(a_1, a_2, \cdots a_N; x), \tag{3.3-1}$$

and let the a's be random variables. For a given set of a's, this equation gives a curve of y versus x. Since the a's are random variables we shall call this curve a random curve. Let us select a short interval x_1, $x_1 + dx$, and then draw a batch of a's. The probability that the curve obtained by putting these a's in (3.3-1) will have a zero in x_1, $x_1 + dx$ is

$$dx \int_{-\infty}^{+\infty} |\eta| \, p(0, \eta; x_1) \, d\eta \tag{3.3-2}$$

and the expected number of zeros in the interval (x_1, x_2) is

$$\int_{x_1}^{x_2} dx \int_{-\infty}^{+\infty} |\eta| \, p(0, \eta; x) \, d\eta \tag{3.3-3}$$

In these expressions $p(\xi, \eta; x)$ is the probability density function for the variables

$$\xi = F(a_1, \cdots a_N; x)$$
$$\eta = \frac{\partial F}{\partial x} \tag{3.3-4}$$

Since the a's are random variables so are ξ and η, and their distribution will contain x as a parameter. This is indicated by the notation $p(\xi, \eta; x)$.

These results may be proved in much the same manner as are similar results for the distribution of the maxima of a random curve. This method of proof suffers from the restriction that the a's are required to be bounded.[25] Results equivalent to (3.3-2) and (3.3-3) have been obtained independently by M. Kac.[26] His method of proof has the advantage of not requiring the a's to be bounded.

Here we shall sketch the derivation of a closely related result: The probability that y will pass through zero in x_1, $x_1 + dx$ with positive slope is

$$dx \int_0^{\infty} \eta p(0, \eta; x_1) \, d\eta \tag{3.3-5}$$

We choose dx so small that the portions of all but a negligible fraction of the possible random curves lying in the strip $(x_1, x_1 + dx)$ may be regarded as straight lines. If $y = \xi$ at x_1 and passes through zero for $x_1 < x < x_1 + dx$, its intercept on $y = 0$ is $x_1 - \frac{\xi}{\eta}$ where η is the slope. Thus ξ and η must be of opposite sign and

$$x_1 < x_1 - \frac{\xi}{\eta} < x_1 + dx$$

[25] S. O. Rice, *Amer. Jour. Math.* Vol. 61, pp. 409–416 (1939). However, L. A. MacColl has pointed out to me that a set of sufficient conditions for (3.3-5) to hold is: (a) $p(\xi, \eta; x)$ is continuous with respect to (ξ, η) throughout the $\xi\eta$ plane; and (b) that the integral

$$\int_0^{\infty} p(a\eta, \eta; x_1) \, d\eta$$

converges uniformly with respect to a in some interval $-a_1 \leq a \leq a_2$, where a_1 and a_2 are positive. These conditions are satisfied in all the applications we shall make use of (3.3-5).

[26] M. Kac, *Bull. Amer. Math. Soc.* Vol. 49, pp. 314–320 (1943).

According to the statement of our problem, we are interested only in positive values of η, and we therefore write our inequality as

$$-\eta\, dx < \xi < 0$$

For a given random curve i.e. for a given set of a's ξ and η have the values given by

$$\xi = F(a_1, \cdots a_N ; x_1)$$

$$\eta = \left[\frac{\partial F}{\partial x}\right]_{x=x_1}$$

If these values of ξ and η satisfy our inequality, the curve goes through zero in x_1, $x_1 + dx$. The probability of this happening is[27]

$$\int_0^\infty d\eta \int_{-\eta\, dx}^0 d\xi\, p(\xi, \eta; x_1) = \int_0^\infty [0 - (-\eta\, dx)] p(0, \eta; x_1)\, d\eta$$

where we have made use of the fact that dx is so very small that ξ is effectively zero. The last expression is the same as (3.3–5).

In the same way it may be shown that the probability of y passing through zero in x_1, $x_1 + dx$ with a negative slope is

$$-dx \int_{-\infty}^0 \eta p(0, \eta; x_1)\, d\eta \tag{3.3-6}$$

Expression (3.3–2) is obtained by adding (3.3–5) and (3.3–6).

We are now ready to apply our formulas. We let t, $I(t)$ and φ_n play the roles of x, y, and a_n, respectively, and use

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n), \qquad c_n^2 = 2w(f)\Delta f \tag{2.8-6}$$

[27] MacColl has remarked that the step from the double integral on the left hand side of this equation to the final result (3.3–5) may be made as follows:
It is easily seen that the probability density we are seeking is

$$\left[\frac{d}{d(\Delta x)} \int_0^\infty d\eta \int_{-\eta\Delta x}^0 p(\xi, \eta; x)\, d\xi\right]_{\Delta x=0}$$

Proceeding formally, without regard to conditions validating the analytical operations (for such conditions see the footnote on page 52), we have

$$\frac{d}{d\Delta x} \int_0^\infty d\eta \int_{-\eta\Delta x}^0 p(\xi, \eta; x)\, d\xi = \int_0^\infty \eta p(-\eta\Delta x, \eta; x)\, d\eta$$

and hence the required probability density is

$$\int_0^\infty \eta p(0, \eta; x)\, d\eta$$

The first step is to find the probability density function of the two random variables

$$\xi = \sum_{n=1}^{N} c_n \cos(\omega_n t_1 - \varphi_n)$$
$$\eta = I'(t_1) = -\sum_{n=1}^{N} c_n \omega_n \sin(\omega_n t_1 - \varphi_n) \qquad (3.3\text{-}7)$$

where the prime denotes differentiation with respect t. From section 2.10

$$\mu_{11} = \overline{\xi^2} = \psi_0$$
$$\mu_{22} = \overline{\eta^2} = \sum_{n=1}^{N} c_n^2 \omega_n^2 \overline{\sin^2(\omega_n t_1 - \varphi_n)}$$
$$= \sum_{n=1}^{N} (2\pi f_n)^2 w(f_n)\Delta f$$
$$\to 4\pi^2 \int_0^\infty f^2 w(f)\, df = -\psi_0''$$
$$\mu_{12} = \overline{\xi\eta} = -\sum_{n=1}^{N} c_n^2 \omega_n \overline{\cos(\omega_n t_1 - \varphi_n)\sin(\omega_n t_1 - \varphi_n)}$$
$$= 0$$

The expression for μ_{22} arises from (2.1-6) by differentiation. In this expression ψ_0'' denotes the second derivative of $\psi(\tau)$ with respect to τ at $\tau = 0$:

$$\psi''(\tau) = -4\pi^2 \int_0^\infty f^2 w(f) \cos 2\pi f\tau\, df \qquad (3.3\text{-}8)$$

Hence the probability density is

$$p(\xi, \eta; t) = \frac{[-\psi_0\psi_0'']^{-1/2}}{2\pi} \exp\left[-\frac{\xi^2}{2\psi_0} + \frac{\eta^2}{2\psi_0''}\right] \qquad (3.3\text{-}9)$$

where ψ_0'' is negative. It will be observed that the expression on the right is independent of t. Hence the probability of having a zero in $t_1, t_1 + dt$,

$$dt \int_{-\infty}^{+\infty} |\eta| \frac{[-\psi_0\psi_0'']^{-1/2}}{2\pi} e^{\eta^2/2\psi_0''}\, d\eta = \frac{dt}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} \qquad (3.3\text{-}10)$$

which follows from (3.3-3), is independent of t.

The expected number of zeros per second, which may be obtained from (3.3-3) by integrating (3.3-10) over an interval of one second, is

$$\frac{1}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} = 2\left[\frac{\int_0^\infty f^2 w(f)\, df}{\int_0^\infty w(f)\, df}\right]^{1/2} \qquad (3.3\text{-}11)$$

For an ideal band pass filter whose pass band extends from f_a to f_b the expected number of zeros per second is

$$2\left[\frac{1}{3}\frac{f_b^3 - f_a^3}{f_b - f_a}\right]^{1/2} \qquad (3.3\text{-}12)$$

When f_a is zero this becomes $1.155\, f_b$ and when f_a is very nearly equal to f_b it approaches $f_b + f_a$.

In a recent paper M. Kac[28] has given a result which, after a slight generalization, leads to

$$e^{-I^2/2\psi_0}\frac{1}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} dt \qquad (3.3\text{-}13)$$

for the probability that the noise current will pass through the value I with positive slope during the interval $t,\ t + dt$. The expected number of such passages per second is

$$e^{-I^2/2\psi_0} \times [\tfrac{1}{2} \text{ the expected number of zeros per second}] \qquad (3.3\text{-}14)$$

The expression (3.3-13) may also be derived from analogue of (3.3-5) obtained by replacing the zero in $p(0, \eta; x_1)$ by y.

In some cases the integral

$$\psi_0'' = -4\pi^2 \int_0^\infty f^2 w(f)\, df$$

does not converge.

An example occurs when we apply a broad band noise voltage to a resistance and condenser in series. The power spectrum of the voltage across the condenser is of the form

$$w(f) = \frac{1}{f^2 + a^2} \qquad (3.3\text{-}15)$$

Although ψ_0'' is infinite, ψ_0 is finite and equal to $\pi/2a$. A straightforward substitution in our formula (3.3-11) gives infinity as the expected number of zeros per second.

Some light is thrown on this breakdown of our formula when we consider a noise current consisting of two bands of noise. One band is confined to relatively low frequencies, and its power spectrum will be denoted by $w_1(f)$. The other band is very narrow and is centered at the relatively high frequency f_2. The complete power spectrum of our noise is then

$$w(f) = w_1(f) + A^2\delta(f - f_2)$$

[28] On the Distribution of Values of Trigonometric Sums with Linearly Independent Frequencies, *Amer. Jour. Math.*, Vol. LXV, pp 609-615, (1943).

where the unit impulse function δ is used to represent the very narrow band. The power spectrum of the narrow band is approximately the same as that of the wave $A\sqrt{2}\cos 2\pi f_2 t$.

The integrals occurring in our formula are

$$\int_0^\infty w(f)\,df = \int_0^\infty w_1(f)\,df + A^2$$
$$= W + A^2$$
$$\int_0^\infty w(f)f^2\,df = \int_0^\infty f^2 w_1(f)\,df + A^2 f_2^2$$
$$= U + A^2 f_2^2$$

We suppose that A and f_2 are such that

$$W \gg A^2$$
$$U \ll A^2 f_2^2 .$$

Then our formula (3.3-11) gives us the expected number of zeros

$$2\,\frac{Af_2}{W^{1/2}}$$

We may give a qualitative explanation of this formula if we regard our noise current as composed of a small component

$$I_2 = 2^{1/2} A \cos 2\pi f_2 t$$

due to the narrow band superposed on a large, slowly varying component due to the lower band. Since the r.m.s. value of the second component is $W^{1/2}$ we may assign it a representative frequency f_1 and write it approximately as

$$I_1 = (2W)^{1/2} \cos 2\pi f_1 t$$

The zeros of the noise current are clustered around the zeros of the second wave. Near such a zero

$$I_1 = \pm (2W)^{1/2} 2\pi f_1 \Delta t$$

where Δt is the distance from the zero. The oscillations of I_1 produce zeros when $|I_1|$ is less than the amplitude of I_2 or when

$$A > W^{1/2} 2\pi f_1 |\Delta t|$$

and the interval over which zeros are produced is given by

$$2\Delta t = \frac{AW^{-1/2}}{\pi f_1}$$

The number of zeros is this multiplied by $2f_2$. Since there are $2f_1$ such intervals per second the number of zeros per second is

$$\frac{4}{\pi} AW^{-1/2} f_1$$

This differs from the result given by our formula by a factor of $2/\pi$. This discrepancy is due to our representing the two bands by the sine waves I_1 and I_2.

From this example we obtain the picture that when the integral for ψ_0 converges corresponding to $A \rightarrow 0$, while at the same time the integral for ψ_0'' diverges, corresponding to $f_2 \rightarrow \infty$ in such a way that $Af_2 \rightarrow \infty$, the noise current behaves something like a continuous function which has no derivative. It seems that for physical systems the integrals will always converge since parasitic effects will have the effect of making $w(f)$ tend to zero rapidly enough. The frequency which represents the region where this occurs is of the order of the frequency of the microscopic wiggles.

So far we have been considering the formulas of this section in the most favorable light possible. There are experiments which indicate the possibility of the formulas breaking down in some cases. Prof. Uhlenbeck has pointed out that if a very broad band fluctuation current be forced[29] to flow through a circuit consisting of a condenser, C, in parallel with a series combination of inductance, L, and resistance, R, equation (3.3–11) says that the expected number of zeros per second of the current, I, flowing through R (and L) is independent of R. It is simply $\frac{1}{\pi}(LC)^{-1/2}$. The differential equation for I is the same as that which governs the Brownian motion of a mirror suspended in a gas[30], the gas pressure playing the role of R. Curves are available for this motion and it is seen that their character depends greatly upon the pressure[31]. Unfortunately, it is difficult to tell from the curves whether the expected number of zeros is independent of the pressure. The differences between the curves for various pressures indicates that there may be some dependence*.

3.4 The Distribution of Zeros

The problem of determining the distribution function for the distance between two successive zeros seems to be quite difficult and apparently

[29] For example, by putting the circuit in series with a diode.

[30] This problem in Brownian motion is discussed by G. E. Uhlenbeck and S. Goudsmit, *Phys., Rev.*, 34 (1929), 145–151.

[31] E. Kappler, *Annalen d. Phys.*, 11 (1931) 233–256.

* Since this was written M. Kac and H. Hurwitz have studied the problem of the expected number of zeros using quite a different method of approach which employs the "shot-effect" representation (Sec. 3.11). Their results confirm the correctness of (3.3–11) when the integrals converge. When the integrals diverge the average number of electrons, per sec. producing the shot effect must be considered.

nobody has as yet given a satisfactory solution. Here we shall give some results which are related to the general problem and which give an idea of the form of the distribution for the region of small spacings between the zeros.

We shall show (in the work starting with equation (3.4-12)) that the probability of the noise current, I, passing through zero in the interval $\tau, \tau + d\tau$ with a negative slope, when it is known that I passes through zero at $\tau = 0$ with a positive slope, is

$$\frac{d\tau}{2\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}\left[\frac{M_{23}}{H}\right](\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H \cot^{-1}(-H)] \tag{3.4-1}$$

where M_{22} and M_{23} are the cofactors of $\mu_{22} = -\psi_0''$ and $\mu_{23} = -\psi_\tau''$ in the matrix

$$M = \begin{bmatrix} \psi_0 & 0 & \psi_\tau' & \psi_\tau \\ 0 & -\psi_0'' & -\psi_\tau'' & -\psi_\tau' \\ \psi_\tau' & -\psi_\tau'' & -\psi_0'' & 0 \\ \psi_\tau & -\psi_\tau' & 0 & \psi_0 \end{bmatrix}, \tag{3.4-2}$$

$$H = M_{23}[M_{22}^2 - M_{23}^2]^{-1/2}.$$

We choose $0 \leq \cot^{-1}(-H) \leq \pi$, the value π being taken at $\tau = 0$, and the value $\pi/2$ being approached as $\tau \rightarrow \infty$. It should be remembered that we are writing the arguments of the correlation functions as subscripts, e.g., $-\psi_\tau''$ is really

$$-\psi''(\tau) = 4\pi^2 \int_0^\infty f^2 w(f) \cos 2\pi f\tau \, df \tag{3.3-8}$$

As τ becomes larger and larger the behavior of I at τ is influenced less and less by the fact that it goes through zero with a positive slope at $\tau = 0$. Hence (3.4-1) should approach the probability that, for any interval of length $d\tau$ chosen at random, I will go through zero with a negative slope. Because of symmetry, this is half the probability that it will go through zero. Thus (3.4-1) should approach, from (3.3-10),

$$\frac{d\tau}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} \tag{3.4-3}$$

as $\tau \rightarrow \infty$. It actually does this since M approaches a diagonal matrix and both M_{23} and H approach zero with $M_{23}/H \rightarrow M_{22} \rightarrow -\psi_0^2\psi_0''$. For a low pass filter cutting off at f_b (3.4-3) is

$$d\tau f_b 3^{-1/2} \tag{3.4-4}$$

The behavior of (3.4-1) as $\tau \rightarrow 0$ is quite a bit more difficult to work out.

M_{22} and M_{23} go to zero as τ^4, $M_{22}^2 - M_{23}^2$ as τ^{10}, and consequently H goes to infinity as τ^{-1}. The final result is that (3.4-1) approaches

$$d\tau \frac{\tau}{8}\left[\frac{\psi_0\psi_0^{(4)} - \psi_0''^2}{-\psi_0\psi_0''}\right] \tag{3.4-5}$$

as $\tau \to 0$, assuming $\psi^{(4)}$ exists. Here the superscript (4) indicates the fourth derivative at $\tau = 0$,

$$\psi_0^{(4)} = 16\pi^4 \int_0^\infty f^4 w(f)\, df \tag{3.4-6}$$

For a low pass filter cutting off at f_b (3.4-5) is

$$d\tau \frac{\tau}{30}(2\pi f_b)^2 \tag{3.4-7}$$

When (3.4-1) is applied to a low pass filter, it turns out that instead of τ the variable

$$\varphi = 2\pi f_b \tau, \qquad d\varphi = 2\pi f_b\, d\tau \tag{3.4-8}$$

is more convenient to handle. Thus, if we write (3.4-1) as $p(\varphi)\, d\varphi$, it follows from (3.4-4) and (3.4-7) that

$$\begin{aligned} p(\varphi) &\to \frac{1}{2\pi\sqrt{3}} = .0919 \quad \text{as} \quad \varphi \to \infty \\ p(\varphi) &\to \frac{\varphi}{30} \quad \text{as} \quad \varphi \to \infty \end{aligned} \tag{3.4-9}$$

$p(\varphi)$ has been computed and plotted on Fig. 1 as a function of φ for the range 0 to 9. From the curve and the theory it is evident that beyond 9 $p(\varphi)$ oscillates about 0.0919 with ever decreasing amplitude.

We may take $p(\varphi)\, d\varphi$ to be the probability that I goes through zero in $\varphi, \varphi + d\varphi$, when it is known that I goes through zero at $\varphi = 0$ with a slope opposite to that at φ. $p(\varphi)\, d\varphi$ exceeds the probability that I goes through zero at $\varphi = 0$ and in $\varphi, \varphi + d\varphi$ with no zeros in between. This is because $p(\varphi)\, d\varphi$ includes all curves of the latter class and in addition those which may have an even number of zeros between 0 and φ. From this it follows that the curve giving the probability density of the intervals between zeros must be underneath the curve of $p(\varphi)$.

A partial check on the curve for $p(\varphi)$ may be obtained by comparing it with a probability density function obtained experimentally by M. E. Campbell for the intervals between 754 successive zeros. He passed thermal noise through a band pass filter, the lower cutoff being around 200 cps and the upper cutoff being around 3000 cps. The upper cutoff was rather gradual and it is difficult to assign a representative value. The crosses on figure 1 are obtained from his data when we assume that his filter behaves like a low pass filter with a cutoff at $f_b = 2850$, this choice being made in order to make the maximum of his curve coincide with that of $p(\varphi)$.

It is seen that some of the crosses lie above $p(\varphi)$. This is probably due to the fact that the actual filter differs somewhat from the assumed low pass filter.

On Fig. 1 there is also plotted a function closely related to (3.4-1). It is the low pass filter form of the following: The probability of I passing

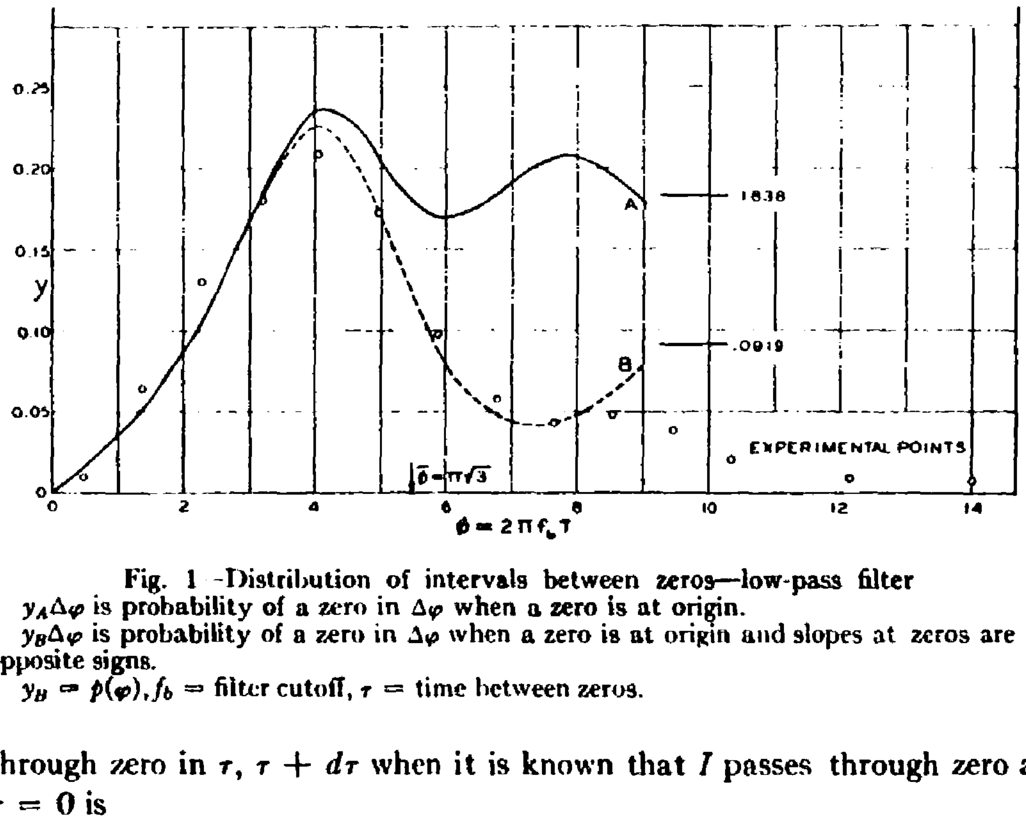

Fig. 1 -Distribution of intervals between zeros—low-pass filter
$y_A \Delta\varphi$ is probability of a zero in $\Delta\varphi$ when a zero is at origin.
$y_B \Delta\varphi$ is probability of a zero in $\Delta\varphi$ when a zero is at origin and slopes at zeros are of opposite signs.
$y_B = p(\varphi), f_b$ = filter cutoff, τ = time between zeros.

through zero in τ, $\tau + d\tau$ when it is known that I passes through zero at $\tau = 0$ is

$$\frac{d\tau}{\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}\left[\frac{M_{23}}{H}\right](\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H \tan^{-1} H] \qquad (3.4\text{-}10)$$

where the notation is the same as in (3.4-1) and $-\frac{\pi}{2} \leq \tan^{-1} H \leq \frac{\pi}{2}$. This curve should always lie above $p(\varphi)$ and the small difference between the curves out to $\varphi = 4$ indicates that the true distribution of zeros is given closely by $p(\varphi)$ out to this point.

When (3.4-1) is applied to a relatively narrow band pass filter or some similar device we may make some approximations and obtain an expression somewhat simpler than (3.4-1). As a guide we consider our usual ideal band pass filter whose range extends from f_a to f_b. The correlation function is given by (3.2–5).

$$\psi_\tau = \frac{w_0}{\pi\tau} \sin \pi\tau(f_b - f_a) \cos \pi\tau(f_b + f_a)$$
$$\psi_0 = w_0(f_b - f_a) \qquad (3.2\text{–}5)$$

From physical considerations we know that in a narrow filter most of the distances between zeros will be nearly equal to

$$\tau_1 = \frac{1}{f_b + f_a}$$

i.e., nearly equal to the distance between the zeros of a sine wave having the mid-band frequency. We therefore expect (3.4–1) to have a peak very close to τ_1. We also expect peaks at $3\tau_1$, $5\tau_1$ etc. but we shall not consider these. We wish to examine the behavior of (3.4–1) near τ_1.

It turns out that M_{23} is nearly equal to M_{22} so that H is large and (3.4–1) becomes approximately

$$\frac{d\tau}{2}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2} \frac{M_{23}}{[\psi_0^2 - \psi_\tau^2]^{3/2}}$$

where τ is near τ_1.

In order to see that M_{23} is nearly equal to M_{22} we use the expressions

$$M_{22} = -\psi_0''(\psi_0^2 - \psi_\tau^2) - \psi_0\psi_\tau'^2$$
$$M_{23} = \psi_\tau''(\psi_0^2 - \psi_\tau^2) + \psi_\tau\psi_\tau'^2$$
$$M_{22} + M_{23} = (\psi_0 - \psi_\tau)[(\psi_0 + \psi_\tau)(\psi_\tau'' - \psi_0'') - \psi_\tau'^2]$$
$$= (\psi_0 - \psi_\tau)[B + C]$$
$$M_{22} - M_{23} = (\psi_0 + \psi_\tau)[(\psi_0 - \psi_\tau)(-\psi_\tau'' - \psi_0'') - \psi_\tau'^2]$$
$$= (\psi_0 + \psi_\tau)[-B + C]$$
$$B = \psi_0\psi_\tau'' - \psi_\tau\psi_0''$$
$$C = -\psi_0\psi_0'' + \psi_\tau\psi_\tau'' - \psi_\tau'^2$$

From (3.2–5) it is seen that ψ_τ may be written as

$$\psi_\tau = A \cos \beta\tau, \qquad \beta = \pi(f_b + f_a)$$

where $\beta\tau_1 = \pi$ and A is a function of τ which varies slowly in comparison with $\cos \beta\tau$. We see that near τ_1, ψ_τ is nearly equal to $-\psi_0$. Likewise

ψ'_τ hovers around zero and ψ''_τ is nearly equal to $-\psi''_0$. Differentiating with respect to τ gives

$$\psi'_\tau = A' \cos \beta\tau - A\beta \sin \beta\tau$$

$$\psi''_\tau = (A'' - A\beta^2) \cos \beta\tau - 2A'\beta \sin \beta\tau$$

$$\psi''_0 = A''_0 - A_0\beta^2, \qquad \psi_0 = A_0$$

where A_0 and A''_0 are the values of A and its second derivative at τ equal to zero. These lead to

$$B = (A_0A'' - AA''_0) \cos \beta\tau - 2A_0A'\beta \sin \beta\tau$$

$$C = (AA'' - A'^2) \cos^2 \beta\tau - A_0A''_0 + (A_0^2 - A)^2\beta^2$$

We wish to show that $C + B$ and $C - B$ are of the same order of magnitude. If we can do this, it follows that $M_{22} - M_{23}$ is much smaller than $M_{22} + M_{23}$ since $\psi_0 - \psi_{\tau_1}$ is approximately $2\psi_0$ while $\psi_0 + \psi_{\tau_1}$ is quite small. Consequently we will have shown that M_{23} is nearly equal to M_{22}.

So far we have made no approximations. We now express the slowly varying function A as a power series in τ. Since ψ'_0 and ψ'''_0 must be zero for the type of functions we consider, it follows that

$$A = A_0 + \frac{\tau^2}{2} A''_0 + \cdots$$

$$A' = \tau A''_0 + \cdots$$

$$A'' = A''_0 + \frac{\tau^2}{2} A_0^{(4)} + \cdots$$

where we neglect all powers higher than the second. Multiplication and squaring gives

$$A^2 - A_0^2 = \tau^2 A_0 A''_0$$

$$AA'' - A'^2 = A_0A''_0 + \frac{\tau^2}{2} (A_0A_0^{(4)} - A_0''^2)$$

$$= A_0A''_0 + F$$

$$A_0A'' - AA''_0 = \frac{\tau^2}{2} (A_0A_0^{(4)} - A_0''^2) = F$$

Since, for small τ, A and A'' are nearly equal to A_0 and A''_0, respectively we see that the difference on the left is small relative to $A_0A''_0$, i.e.,

$$|F| << |A_0A''_0|$$

Our expression for B and C become approximately

$$B = F \cos \beta\tau - 2A_0 A_0'' \beta\tau \sin \beta\tau$$

$$C = F \cos^2 \beta\tau - A_0 A_0'' \sin^2 \beta\tau - A_0 A_0'' \beta^2 \tau^2$$

When τ is near τ_1, $\beta\tau$ is approximately π. Hence both $C + B$ and $C - B$ are approximately $-A_0 A_0'' \pi^2$ and are of the same order of magnitude. Consequently M_{22} and M_{23} are both nearly equal and

$$\begin{aligned} M_{23} &= \psi_0 [C + B] \\ &= -A_0^2 A_0'' \pi^2 \end{aligned}$$

When this expression for M_{23} is used our approximation to (3.4–1) gives us the result: If the correlation function is of the form

$$\psi_\tau = A \cos \beta\tau$$

where A is a slowly varying function of τ, the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{d\tau}{2} \frac{a}{[1 + a^2(\tau - \tau_1)^2]^{3/2}}$$

where a is positive and

$$a^2 = \frac{A_0 \beta^2}{-A_0'' \tau_1^2}, \qquad \tau_1 = \frac{\pi}{\beta}$$

For our ideal band pass filter with the pass band $f_b - f_a$,

$$a = \sqrt{3} \frac{(f_b + f_a)^2}{f_b - f_a}, \qquad \tau_1 = \frac{1}{f_b + f_a}$$

and the average value of $|\tau - \tau_1|$ is a^{-1}. Thus

$$\frac{\text{ave.} |\tau - \tau_1|}{\tau_1} = \frac{1}{a\tau_1} = \frac{f_b - f_a}{\sqrt{3}\,(f_b + f_a)} = \frac{1}{2\sqrt{3}} \frac{\text{band width}}{\text{mid-frequency}}$$

When the correlation function cannot be put in the form assumed above but still behaves like a sinusoidal wave with slowly varying amplitude we may use our first approximation to (3.4–1). Thus, the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{b\, d\tau}{[\psi_0^2 - \psi_\tau^2]^{3/2}}$$

when τ lies near τ_1 where τ_1 is the smallest value of τ which makes ψ_τ approximately equal to $-\psi_0$. This probability is supposed to approach zero rapidly as τ departs from τ_1, and b is chosen so that the integral over the effective region around τ_1 is unity.

It seems to be especially difficult to get an expression for the distribution of zeros for large spacing. One method, suggested by Prof. Goudsmit, is to amend the conditions leading to (3.4–1) by adding conditions that I be positive at equally spaced points along the time axis between 0 and τ. This leads to integrals which are hard to evaluate. For one point between 0 and τ the integral is of the form (3.5–7).

Another method of approach is to use the method of "in and exclusion" of zeros between 0 and τ. Consider the class of curves of I having a zero at $\tau = 0$. Then, in theory, our methods will allow us to compute the functions $p_0(\tau)$, $p_1(r, \tau)$, $p_2(r, s, \tau)$, associated with this class where

$p_0(\tau)\, d\tau$ is probability of curve having zero in $d\tau$

$p_1(r, \tau)\, d\tau\, dr$ is probability of curve having zeros in $d\tau$ and dr

$p_2(r, s, \tau)\, d\tau\, dr\, ds$ is probability of curve having zeros in $d\tau$, dr, and ds

In fact $p_0(\tau)\, d\tau$ is expression (3.4–10). The method of in and exclusion then leads to an expression for $P_0(\tau)\, d\tau$, the probability of having a zero at 0 and a zero in τ, $\tau + d\tau$ but none between 0 and τ. It is

$$P_0(\tau) = p_0(\tau) - \frac{1}{1!}\int_0^\tau p_1(r, \tau)\, dr + \frac{1}{2!}\int_0^\tau\int_0^\tau p_2(r, s, \tau)\, dr\, ds - \frac{1}{3!}\int_0^\tau\int_0^\tau\int_0^\tau p_3(r, s, t, \tau)\, dr\, ds\, dt + \dots \qquad (3.4\text{–}11)$$

Here again we run into difficult integrals. Incidentally, (3.4–11) may be checked for events occurring independently at random. Thus if $\nu\, d\tau$ is the probability of an event happening in $d\tau$, then, if ν is a constant and the events are independent, we have p_0, p_1, p_2, $\cdots$ given by ν, ν^2, ν^3, $\cdots$. From (3.4–11) we obtain the known result $P_0(\tau) = \nu e^{-\nu\tau}$.

We shall now derive (3.4–1). The work is based upon a generalization of (3.3–5): If y is a random curve described by (3.3–1), the probability that y will pass through zero in x_1, $x_1 + dx_1$ with a positive slope and through zero in x_2, $x_2 + dx_2$ with a negative slope is

$$-dx_1\, dx_2 \int_0^{+\infty} d\eta_1 \int_{-\infty}^{0} d\eta_2\, \eta_1 \eta_2\, p(0, \eta_1, x_1;\, 0, \eta_2, x_2) \qquad (3.4\text{–}12)$$

where $p(\xi_1, \eta_1, x_1;\, \xi_2, \eta_2, x_2)$ is the probability density function for the four random variables

$$\xi_i = F(a_1, a_2, \cdots, a_N;\, x_i)$$

$$\eta_i = \left[\frac{\partial F}{\partial x}\right]_{x=x_i}, \qquad i = 1, 2.$$

The x_1 and x_2 play the role of parameters in (3.4–12). This result may be established in much the same way as (3.3–5).

When we identify F with one of our representations, (2.8–1) or (2.8–6), of the noise current $I(t)$ it is seen that p is normal in four dimensions. We may obtain the second moments directly from this representation, as has been done in the equations just below (3.3–7). The same results may be obtained from the definition of $\psi(\tau)$, and for the sake of variety we choose this second method. We set $x_1 = t_1$, $x_2 = t_1 + \tau$. Then

$$\overline{\xi_1^2} = \overline{\xi_2^2} = \overline{I^2(t)} = \psi_0$$

$$\overline{\xi_1 \xi_2} = \overline{I(t)I(t + \tau)} = \psi_\tau \tag{3.4–13}$$

$$\overline{\eta_1 \eta_2} = \overline{\left(\frac{\partial I}{\partial t}\right)_t \left(\frac{\partial I}{\partial t}\right)_{t+\tau}} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T} \int_0^T I'(t + \tau)I'(t)\, dt$$

where primes denote differentiation with respect to the arguments. Integrating by parts:

$$\int_0^T I'(t + \tau)\, dI(t) = [I'(t + \tau)I(t)]_0^T - \int_0^T I''(t + \tau)I(t)\, dt$$

We assume that I and its derivative remains finite so that the integrated portion vanishes, when divided by T, in the limit. Since

$$I''(t + \tau) = \frac{\partial^2}{\partial \tau^2} I(t + \tau)$$

we have

$$\overline{\eta_1 \eta_2} = -\frac{\partial^2}{\partial \tau^2} \psi(\tau) = -\psi_\tau''$$

Setting $\tau = 0$ gives

$$\overline{\eta_1^2} = \overline{\eta_2^2} = -\psi_0''$$

in agreement with the value of μ_{22} obtained from (3.3–7). In the same way

$$\overline{\xi_1 \eta_2} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T} \int_0^T I'(t + \tau)I(t)\, dt = \frac{\partial}{\partial \tau} \psi(\tau)$$
$$= \psi_\tau'$$

$$\overline{\xi_2 \eta_1} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T} \int_0^T I'(t)I(t + \tau)\, dt$$
$$= \text{“} \quad (-)\frac{1}{T} \int_0^T I'(t + \tau)I(t)\, dt$$
$$= -\psi_\tau'$$

where we have integrated by parts in getting $\overline{\xi_2\eta_1}$. Setting $\tau = 0$ and using $\psi_0' = 0$ gives

$$\overline{\xi_1 \eta_1} = \overline{\xi_2 \eta_2} = 0$$

In order to obtain the matrix M of the second moments μ_{rs} in a form fairly symmetrical about its center we choose the 1, 2, 3, 4 order of our variables to be ξ_1, η_1, η_2, ξ_2. From equations (3.4–13) etc. it is seen that this choice leads to the expression (3.4–2) for M.

When we put ξ_1 and ξ_2 equal to zero, we obtain for the probability density function in (3.4–12) the expression

$$\frac{|M|^{-1/2}}{4\pi^2} \exp\left[-\frac{1}{2|M|}(M_{22}\eta_1^2 + 2M_{23}\eta_1\eta_2 + M_{33}\eta_2^2)\right]$$

Because of the symmetry of M, M_{22} is equal to M_{33}. When, in the integral (3.4–12) we make the change of variable

$$x = \left[\frac{M_{22}}{2|M|}\right]^{1/2}\eta_1, \qquad y = -\left[\frac{M_{22}}{2|M|}\right]^{1/2}\eta_2$$

we obtain

$$\frac{dx_1\, dx_2}{\pi^2}\frac{|M|^{3/2}}{M_{22}^2}\int_0^\infty x\, dx \int_0^\infty dy\, y e^{-x^2-y^2+2(M_{23}/M_{22})xy}$$

The double integral may be evaluated by (3.5–4). Let

$$\varphi = \cos^{-1}\left(-\frac{M_{23}}{M_{22}}\right) = \cot^{-1}(-H), \qquad H = M_{23}[M_{22}^2 - M_{23}^2]^{-1/2}$$

where H is the same as that given in (3.4–2). Our expression now becomes

$$\frac{dx_1\, dx_2}{4\pi^2}\frac{|M|^{3/2}}{M_{22}^2 - M_{23}^2}[1 + H\cot^{-1}(-H)]$$

From a property of determinants

$$M_{22}M_{33} - M_{23}^2 = |M|(\psi_0^2 - \psi_\tau^2)$$

Using this to eliminate $|M|$ and dividing by

$$\frac{dx_1}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2}$$

which, from (3.3–10), is the probability of going through zero in x_1, $x_1 + dx_1$ with positive slope, gives the probability of going through zero in dx_2 with negative slope when it is known that I goes through zero at x_1 with positive slope:

$$\frac{dx_2}{2\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}[M_{22}^2 - M_{23}^2]^{1/2}(\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H \cot^{-1}(-H)]$$

This is the same as (3.4–1).

The expression (3.4–10) is the same as the probability of I going through zero in $d\tau$ when it is known that I goes through zero at the origin with positive slope. This second probability may be obtained from (3.4–1) by adding the probability that I goes through $d\tau$ with positive slope when it is known to go through zero with positive slope. Thus we must add the expression containing the integral in which the integration in both η_1 and η_2 run from 0 to ∞. In terms of x and y this integral is

$$\int_0^\infty x\,dx \int_0^\infty dy\, y e^{-x^2-y^2-2(M_{23}/M_{22})xy}$$

This is equivalent to a change in the sign of M_{23} and hence of H. After this addition we must consider

$$\begin{aligned} 1 + H \cot^{-1}(-H) &+ 1 - H \cot^{-1} H \\ &= 2 + H[\cot^{-1}(-H) - \cot^{-1} H] \\ &= 2 + H[\pi - 2\cot^{-1} H] \\ &= 2[1 + H \tan^{-1} H] \end{aligned}$$

and this leads to (3.4–10).

3.5 Multiple Integrals

We wish to evaluate integrals of the form

$$J = \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-x_1^2-2ax_1x_2-x_2^2} \qquad (3.5\text{–}1)$$

Our method of procedure is to first reduce the exponent to the sum of squares by a suitable linear change of variable and then change to polar coordinates. This method appears to work also for triple integrals of the same sort, but when it is applied to a four-fold integral, the last integration apparently cannot be put in closed form.

The reduction of the exponent to the sum of squares is based upon the transformation: If*

$$\begin{aligned} x_1 &= h_1 y_1 + h_2 D_{21} y_2 + h_3 D_{31} y_3 + \cdots + h_n D_{n,1} y_n \\ x_2 &= 0 \quad + h_2 D_{22} y_2 + \cdots \qquad\qquad + h_n D_{n,2} y_n \qquad (3.5\text{–}2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= 0 \quad + 0 \quad + \cdots \quad + 0 \quad + h_n D_{n,n} y_n \end{aligned}$$

* T. Fort, Am. Math. Monthly, 43 (1936), pp. 477–481. See also Scott and Mathews, Theory of Determinants, Cambridge (1904), Prob. 63, p. 276.

where $D_0 = 1$, $D_1 = a_{11}$, $D_{r,r} = D_{r-1}$, and D_{rs} is the cofactor of a_{sr} (or of a_{rs} because they are equal) in D_r :

$$D_r = \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1r} \\ a_{12} & a_{22} & \\ a_{1r} & \cdots & a_{rr} \end{vmatrix}, \qquad h_r = [D_{r-1}D_r]^{-1/2},$$

then, if none of the D_r's is zero,

$$\sum_1^n a_{rs} x_r x_s = y_1^2 + y_2^2 + \cdots + y_n^2$$

From (3.5–2); the Jacobian $\partial(x_1, \cdots x_n)/\partial(y_1, \cdots y_n)$ is equal to $D_n^{-1/2}$.

Applying our transformation to the exponent:

$$x_1 = y_1 - aD_2^{-1/2}y_2$$

$$x_2 = 0 + D_2^{-1/2}y_2$$

$$D_2 = 1 - a^2$$

Since x_2 runs from 0 to ∞ so must y_2. The expression for x_1 shows that y_1 runs from $a\, D_2^{-1/2}y_2$ to ∞. The integral is therefore

$$J = D_2^{-1/2} \int_0^\infty dy_2 \int_{aD_2^{-1/2}y_2}^\infty e^{-y_1^2-y_2^2}\, dy_1$$

We now change to polar coordinates:

$$y_1 = \rho \cos\theta \qquad dy_1\, dy_2 = \rho\, d\rho\, d\theta$$

$$y_2 = \rho \sin\theta$$

$$y_2 \geq 0 \text{ gives } 0 \leq \theta \leq \pi$$

$$y_1 \geq aD_2^{-1/2}y_2 \text{ gives } \cot\theta \geq aD_2^{-1/2}$$

and obtain

$$J = D_2^{-1/2} \int_0^{\cot^{-1} aD_2^{-1/2}} d\theta \int_0^\infty \rho e^{-\rho^2}\, d\rho$$

$$= \tfrac{1}{2}D_2^{-1/2} \cot^{-1}(aD_2^{-1/2})$$

where the arc-cotangent lies between 0 and π. This may be written in the simpler form

$$J = \tfrac{1}{2}(1 - a^2)^{-1/2} \cos^{-1} a = \tfrac{1}{2}\varphi \csc\varphi$$

where

$$a = \cos\varphi,$$

it being understood that $0 \leq \varphi \leq \pi$.

Other integrals may be obtained by differentiation. Thus from

$$\int_0^\infty dx \int_0^\infty dy\, e^{-x^2-y^2-2xy\cos\varphi} = \tfrac{1}{2}\varphi \csc\varphi \tag{3.5-3}$$

we obtain

$$\int_0^\infty dx \int_0^\infty dy\, xy\, e^{-x^2-y^2-2xy\cos\varphi} = \tfrac{1}{4}\csc^2\varphi(1 - \varphi\cot\varphi) \tag{3.5-4}$$

By using the same transformation we may obtain

$$\int_0^\infty dx \int_0^\infty dy\, y e^{-x^2-y^2-2axy} = \frac{\sqrt{\pi}}{4}\frac{1}{1+a} \tag{3.5-5}$$

Of course, we may expand part of the exponential in a power series and integrate termwise but this leads to a series which has to be summed in each particular case:

$$\int_0^\infty dx \int_0^\infty dy\, x^n y^m e^{-x^2-y^2-2axy}$$
$$= \frac{1}{4}\sum_{r=0}^{\infty}\frac{(-2a)^r}{r!}\,\Gamma\left(\frac{n+r+1}{2}\right)\Gamma\left(\frac{m+r+1}{2}\right)$$

If we take $-1 < R(m) < -\frac{1}{2}$, $-1 < R(m) < -\frac{1}{2}$, the series may be summed when $a = 1$. The result stated just below equation (3.8-9) is obtained by continuing m and n analytically.

The same methods will work when the limits are $\pm\infty$. We obtain, when m and n are integers,

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, x^n y^m e^{-x^2-y^2-2xy\cos\varphi}$$
$$= \begin{cases} 0, & n+m \text{ odd} \\ (-)^n\sqrt{\pi}\,\dfrac{\Gamma\left(\dfrac{m+n+1}{2}\right)}{(\sin\varphi)^{n+m+1}} & \\ \qquad F\left(-n, -m; \dfrac{1-n-m}{2}; \dfrac{1-\cos\varphi}{2}\right), & n+m \text{ even} \end{cases} \tag{3.5-6}$$

The hypergeometric function may also be written as

$$F\left(-\frac{n}{2}, -\frac{m}{2}; \frac{1-n-m}{2}; \sin^2\varphi\right)$$

By transformations of this we are led to the following expression for the integral

$$0, \quad n + m \text{ odd},$$

$$\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{(\sin\varphi)^{n+m+1}} F\left(-\frac{n}{2}, -\frac{m}{2}, \frac{1}{2}; \cos^2\varphi\right), \quad m, n \text{ both even},$$

$$-2\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{m}{2}\right)}{(\sin\varphi)^{n+m+1}} \cos\varphi F\left(\frac{1-m}{2}, \frac{1-n}{2}; \frac{3}{2}; \cos^2\varphi\right),$$

$$m, n \text{ odd}$$

As was mentioned earlier, the method used to evaluate the double integrals may also be applied to similar triple integrals. Here we state two results obtained in this way.

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \exp\left[-x^2 - y^2 - z^2 - 2cxy - 2bzx - 2ayz\right]$$
$$= \frac{1}{4}\left[\frac{\pi}{D_3}\right]^{1/2} [\alpha + \beta + \gamma - \pi]$$

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz\, yz \exp\left[-x^2 - y^2 - z^2 - 2cxy - 2bzx - 2ayz\right]$$
$$= \frac{\sqrt{\pi}}{8D_3}\left[\frac{1+a-b-c}{1+a} - \frac{a-bc}{D_3^{1/2}}(\alpha+\beta+\gamma-\pi)\right] \tag{3.5-7}$$

where β and γ are obtained by cyclic permutation of a, b, c from

$$\alpha = \cos^{-1}\frac{a-cb}{(1-c^2)^{1/2}(1-b^2)^{1/2}} = \sin^{-1}\left[\frac{D_3}{(1-c^2)(1-b^2)}\right]^{1/2}$$
$$= \cot^{-1}\frac{a-bc}{D_3^{1/2}}$$

where α, β, γ all lie in the range $0, \pi$ and where

$$D_3 = \begin{vmatrix} 1 & c & b \\ c & 1 & a \\ b & a & 1 \end{vmatrix} = 1 + 2\,abc - a^2 - b^2 - c^2$$

For reference we state the integrals which arise from the definition of the normal distribution given in section (2.9)

$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n \exp\left[-\sum_1^n a_{rs}x_r x_s\right] = \left[\frac{\pi^n}{|a|}\right]^{1/2}$$
$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n\, x_t x_u \exp\left[-\sum^n a_{rs}x_r x_s\right] = \left[\frac{\pi^n}{|a|^3}\right]^{1/2} \frac{A_{tu}}{2} \tag{3.5-8}$$

where the quadratic form is positive definite and $|a|$ is its determinant. A_{tu} is the cofactor of a_{tu}. Incidentally, these may be regarded as special cases of

$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n f\left(\sum_1^n a_{rs} x_r x_s\right) F\left(\sum_1^n b_r x_r\right)$$

$$= \frac{2}{\Gamma\left(\frac{n-1}{2}\right)} \left[\frac{\pi^{n-1}}{|a|}\right]^{1/2} \int_{-\infty}^{+\infty} dx \int_0^{\infty} dy\, y^{n-2} f(x^2 + y^2) \qquad (3.5\text{-}9)$$

$$F\left\{x\left[\frac{\sum_1^n A_{rs} b_r b_s}{|a|}\right]^{1/2}\right\},$$

which is a generalization of a result given by Schlömilch.*

3.6 Distribution of Maxima of Noise Current

Here we shall use a result similar to those used in sections 3.3 and 3.4. Let y be a random curve given by (3.3-1),

$$y = F(a_1 \cdots a_N ; x). \qquad (3.3\text{-}1)$$

If suitable conditions are satisfied, the probability that y has a maximum in the rectangle $(x_1, x_1 + dx_1, y_1, y_1 + dy_1)$, dx_1 and dy_1 being of the same order of magnitude, is[32]

$$-dx_1\, dy_1 \int_{-\infty}^{0} p(y_1, 0, \zeta)\zeta\, d\zeta \qquad (3.6\text{-}1)$$

and the expected number of maxima of y in $a \le x \le b$ is obtained by integrating this expression over the range $-\infty \le y_1 \le \infty$, $a \le x_1 \le b$. $p(\xi, \eta, \zeta)$ is the probability density function for the random variables

$$\xi = F(a_1, \cdots, a_N ; x_1)$$

$$\eta = \left(\frac{\partial F}{\partial x}\right)_{x=x_1} \qquad (3.6\text{-}2)$$

$$\zeta = \left(\frac{\partial^2 F}{\partial x^2}\right)_{x=x_1}$$

* Höheren Analysis, Braunschweig (1879), Vol. 2, p. 494, equ. (29).

[32] *Am. Jour. Math.*, Vol. 61 (1939) 409–416. A similar problem has been studied by E. L. Dodd, The Length of the Cycles Which Result From the Graduation of Chance Elements, Ann. Math. Stat., Vol. 10 (1939) 254–264. He gives a number of references to the literature dealing with the fluctuations of time series.

In our application of this result we replace x and y by t and I as before. Then

$$\xi = I = \sum_{1}^{N} c_n \cos(\omega_n t - \varphi_n)$$

$$\eta = I'$$

$$\zeta = I''$$

where the primes denote differentiation with respect to t. According to the central limit theorem the distribution of ξ, η, ζ approaches a normal law. The second moments defining this law may be obtained either from the above definitions of ξ, η, ζ, or may be obtained from the correlation function as was done in the work following equation (3.4–13).

$$\overline{\xi^2} = \psi_0, \qquad \overline{\eta^2} = -\psi_0'', \qquad \overline{\xi\eta} = 0$$

$$\overline{\eta\zeta} = \overline{I'(t)I''(t)} = \underset{T\to\infty}{\text{Limit}} \frac{1}{T}\int_0^T I'(t)I''(t)\,dt$$

$$= \underset{T\to\infty}{\text{Limit}} \frac{1}{2T}[I'^2(T) - I'^2(0)] = 0$$

$$\overline{\xi\zeta} = \underset{T\to\infty}{\text{Limit}} \frac{1}{T}\int_0^T I(t)I''(t)\,dt$$

$$= \underset{\tau\to 0}{\text{Limit}} \frac{\partial^2\psi(\tau)}{\partial\tau^2} = \psi_0''$$

$$\overline{\zeta^2} = \underset{T\to\infty}{\text{Limit}} \frac{1}{T}\int_0^T I''(t)I''(t)\,dt$$

$$= \underset{T\to\infty}{\text{Limit}} \frac{1}{T}\int_0^T I^{(4)}(t)I(t)\,dt$$

$$= \psi_0^{(4)}$$

where the superscript (4) represents the fourth derivative. The matrix M of the moments is thus

$$M = \begin{bmatrix} \psi_0 & 0 & \psi_0'' \\ 0 & -\psi_0'' & 0 \\ \psi_0'' & 0 & \psi_0^{(4)} \end{bmatrix}$$

The determinant $|M|$ and the cofactors of interest are

$$|M| = -\psi_0''(\psi_0\psi_0^{(4)} - \psi_0''^2) \tag{3.6–3}$$

$$M_{11} = -\psi_0''\psi_0^{(4)}, \qquad M_{13} = \psi_0''^2, \qquad M_{33} = -\psi_0''\psi_0$$

The probability density function in (3.6-1) is

$$p(I, 0, \zeta) = (2\pi)^{-3/2} |M|^{-1/2} \exp \left[-\frac{1}{2|M|} (M_{11} I^2 + M_{33} \zeta^2 + 2M_{13} I\zeta) \right] \tag{3.6-4}$$

and when this is put in (3.6-1) and the integration with respect to ζ performed we get

$$dI\, dt \frac{(2\pi)^{-3/2}}{M_{33}} \left[|M|^{1/2} e^{-M_{11} I^2/2|M|} + M_{13} I \left(\frac{\pi}{2M_{33}}\right)^{1/2} e^{-I^2/2\psi_0} \left(1 + \text{erf} \frac{M_{13} I}{[2|M| M_{33}]^{1/2}}\right) \right] \tag{3.6-5}$$

for the probability of a maximum occurring in the rectangle $dI\, dt$. As is mentioned just below expression (3.6-1), the expected number of maxima in the interval t_1, t_2 may be obtained by integrating (3.6-1) from t_1 to t_2 after replacing x by t, and I from $-\infty$ to $+\infty$ after replacing y by I. When we use (3.6-4) it is easier to integrate with respect to I first. The expected number is then

$$-\int_{t_1}^{t_2} dt \frac{M_{11}^{-1/2}}{2\pi} \int_{-\infty}^{0} \zeta \exp\left[-\frac{\zeta^2}{2|M|} \left(M_{33} - \frac{M_{13}^2}{M_{11}} \right) \right] d\zeta$$

$$= (t_2 - t_1) \frac{\psi_0^{(4)}}{2\pi} M_{11}^{-1/2} = \frac{t_2 - t_1}{2\pi} \left[\frac{\psi_0^{(4)}}{-\psi_0''} \right]^{1/2}$$

Hence the expected number of maxima per second is

$$\frac{1}{2\pi} \left[\frac{\psi_0^{(4)}}{-\psi_0''} \right]^{1/2} = \left[\frac{\int_0^\infty f^4 w(f)\, df}{\int_0^\infty f^2 w(f)\, df} \right]^{1/2}, \tag{3.6-6}$$

For a band pass filter, the expected number of maxima per second is

$$\left[\frac{3}{5} \frac{f_b^5 - f_a^5}{f_b^3 - f_a^3} \right]^{1/2} . \tag{3.6-7}$$

where f_b and f_a are the cut-off frequencies. Putting $f_a = 0$ so as to get a low pass filter,

$$f_b \left[\frac{3}{5} \right]^{1/2} = .775 f_b \tag{3.6-8}$$

From (3.6-8) and (3.6-5) we may obtain the probability density function for the maxima in the case of a low pass filter. Thus the probability that a maximum selected at random from the universe of maxima will lie in $I, I + dI$ is

$$\frac{dI}{3\sqrt{2\pi\psi_0}}\left[2e^{-9y^2/8} + \left(\frac{5\pi}{2}\right)^{1/2} ye^{-y^2/2}\left(1 + \text{erf}\, y\left(\frac{5}{8}\right)^{1/2}\right)\right] \tag{3.6-9}$$

where

$$y = \frac{I}{\sqrt{\psi_0}}$$

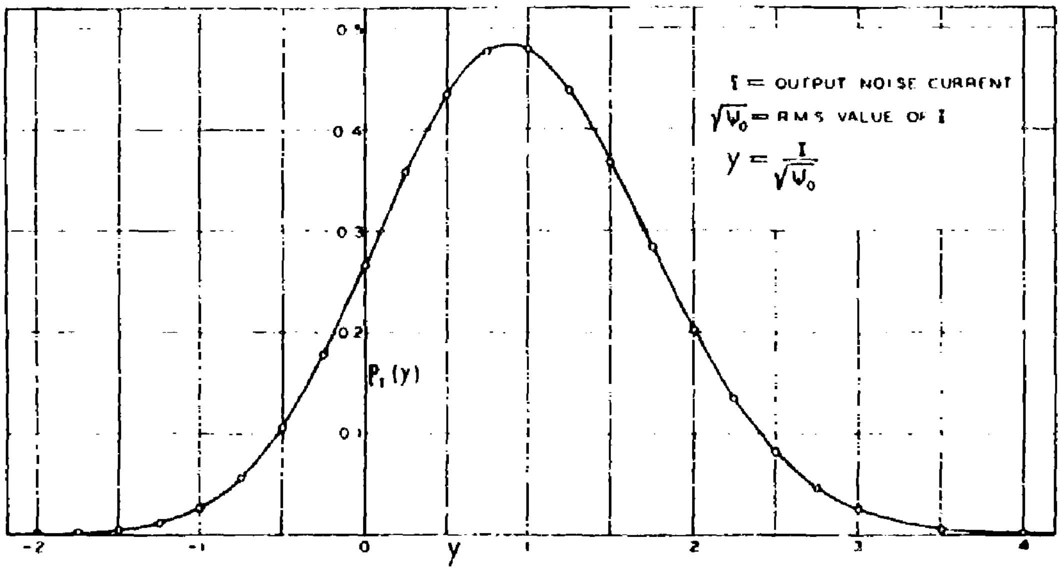

Fig. 2—Distribution of maxima of noise current. Noise through ideal low-pass filter. $\frac{p_I(y)}{\sqrt{\psi_0}}\, dI$ = probability that a maximum of I selected at random lies between I and $I + dI$.

When y is large and positive (3.6-9) is given asymptotically by

$$\frac{dI}{\sqrt{\psi_0}}\, \frac{\sqrt{5}}{3}\, ye^{-y^2/2}$$

If we write (3.6-9) as $p_I(y)\, dy$, the probability density $p_I(y)$ of y may be plotted as a function of y. This plot is shown in Fig. 2. The distribution function $P(I_{max} < y\sqrt{\psi_0})$ defined by

$$P(I_{max} < y\sqrt{\psi_0}) = \int_{-\infty}^{y} p_I(y) \cdot dy$$

and which gives the probability that a maximum selected at random is less than a specified $y\sqrt{\psi_0} = I$, is one of the four curves plotted in Fig. 4.

If I is large and positive we may obtain an approximation from (3.6-5). We observe that

$$\frac{M_{11}}{|M|} = \frac{\psi_0^{(4)}}{\psi_0\psi_0^{(4)} - \psi_0''^2} > \frac{1}{\psi_0}$$

so that when I is large and positive

$$e^{-M_{11}I^2/2|M|t} \ll e^{-I^2/2\psi_0}$$

Also, in these circumstances the $1 + \text{erf}$ is nearly equal to two. Thus retaining only the important terms and using the definitions of the M's gives the approximation to (3.6-5):

$$\frac{dI\,dt}{2\pi\psi_0}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} Ie^{-I^2/2\psi_0} \tag{3.6-10}$$

From this it follows that the expected number of maxima per second lying above the line $I = I_1$ is approximately[33] when I_1 is large,

$$\begin{aligned}\frac{1}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} e^{-I_1^2/2\psi_0}\\ = e^{-I_1^2/2\psi_0} \times \tfrac{1}{2}[\text{the expected number of zeros of } I \text{ per second}]\end{aligned} \tag{3.6-11}$$

It is interesting to note that the approximation (3.6-11) for the expected number of maxima above I_1 is the same as the exact expression (3.3-14) for the expected number of times I will pass through I_1 with positive slope.

3.7 Results on the Envelope of the Noise Current

The noise current flowing in the output of a relatively narrow band pass filter has the character of a sine wave of, roughly, the midband frequency whose amplitude fluctuates irregularly, the rapidity of fluctuation being of the order of the band width. Here we study the fluctuations of the envelope of such a wave.

First we define the envelope. Let f_m be a representative midband frequency. Then if

$$\omega_m = 2\pi f_m \tag{3.7-1}$$

the noise current may be represented, see (2.8-6), by

$$\begin{aligned} I &= \sum_{n=1}^{N} c_n \cos(\omega_n t - \omega_m t - \varphi_n + \omega_m t)\\ &= I_c \cos \omega_m t - I_s \sin \omega_m t\end{aligned} \tag{3.7-2}$$

where the components I_c and I_s are

$$\begin{aligned} I_c &= \sum_{n=1}^{N} c_n \cos(\omega_n t - \omega_m t - \varphi_n)\\ I_s &= \sum_{n=1}^{N} c_n \sin(\omega_n t - \omega_m t - \varphi_n)\end{aligned} \tag{3.7-3}$$

[33] This expression agrees with an estimate made by V. D. Landon, *Proc. I. R. E.*, 29 (1941), 50–55. He discusses the number of crests exceeding four times the r.m.s. value of I. This corresponds to $I_1^2 = 16\psi_0$.

The envelope, R, is a function of t defined by

$$R = [I_c^2 + I_s^2]^{1/2} \tag{3.7-4}$$

It follows from the central limit theorem and the definitions (3.7-3) of I_c and I_s that these are two normally distributed random variables. They are independent since $\overline{I_c I_s} = 0$. They both have the same standard deviation, namely the square root of

$$\overline{I_c^2} = \overline{I_s^2} = \overline{I^2} = \int_0^\infty w(f)\, df = \psi_0 \tag{3.7-5}$$

Consequently, the probability that the point (I_c, I_s) lies within the elementary rectangle $dI_c dI_s$ is

$$\frac{dI_c\, dI_s}{2\pi\psi_0} \exp\left[-\frac{I_c^2 + I_s^2}{2\psi_0}\right] \tag{3.7-6}$$

In much of the following work it is convenient to introduce another random variable θ where

$$\begin{aligned} I_c &= R\cos\theta \\ I_s &= R\sin\theta \end{aligned} \tag{3.7-7}$$

Since I_c and I_s are random variables so are R and θ. The differentials are related by

$$dI_c dI_s = R d\theta dR \tag{3.7-8}$$

and the distribution function for R and θ is obtainable from (3.7-6) when the change of variables is made:

$$\frac{d\theta}{2\pi} \frac{R\, dR}{\psi_0} e^{-R^2/2\psi_0} \tag{3.7-9}$$

Since this may be expressed as a product of terms involving R only and θ only, R and θ are independent random variables, θ being uniformly distributed over the range 0 to 2π and R having the probability density[24]

$$\frac{R}{\psi_0} e^{-R^2/2\psi_0} \tag{3.7-10}$$

Expression (3.7-10) gives the probability density for the value of the envelope. Like the normal law for the instantaneous value of I, it depends only upon the average total power

$$\psi_0 = \int_0^\infty w(f)\, df$$

[24] See V. D. Landon and K. A. Norton, *I.R.E. Proc.*, 30 (1942), 425–429.

We now study the correlation between R at time t and its value at some later time $t + \tau$. Let the subscripts 1 and 2 refer to the times t and $t + \tau$, respectively. Then from (3.7-3) and the central limit theorem it follows that the four random variables I_{c1}, I_{s1}, I_{c2}, I_{s2} have a four dimensional normal distribution. This distribution is determined by the second moments

$$\overline{I_{c1}^2} = \overline{I_{s1}^2} = \overline{I_{c2}^2} = \overline{I_{s2}^2} = \psi_0 = \mu_{11}$$

$$\overline{I_{c1} I_{s1}} = \overline{I_{c2} I_{s2}} = 0$$

$$\overline{I_{c1} I_{c2}} = \overline{I_{s1} I_{s2}} = \frac{1}{2} \sum_{n=1}^{N} c_n^2 \cos (\omega_n \tau - \omega_m \tau)$$

$$\to \int_0^\infty w(f) \cos 2\pi(f - f_m)\tau \, df = \mu_{13} \tag{3.7-11}$$

$$\overline{I_{c1} I_{s2}} = -\overline{I_{c2} I_{s1}} = \frac{1}{2} \sum_{n=1}^{N} c_n^2 \sin (\omega_n \tau - \omega_m \tau)$$

$$\to \int_0^\infty w(f) \sin 2\pi(f - f_m)\tau \, df = \mu_{14}$$

The moment matrix for the variables in the order I_{c1}, I_{s1}, I_{c2}, I_{s2} is

$$M = \begin{bmatrix} \psi_0 & 0 & \mu_{13} & \mu_{14} \\ 0 & \psi_0 & -\mu_{14} & \mu_{13} \\ \mu_{13} & -\mu_{14} & \psi_0 & 0 \\ \mu_{14} & \mu_{13} & 0 & \psi_0 \end{bmatrix}$$

and from this it follows that the cofactors of the determinant $|M|$ are

$$\begin{aligned} M_{11} = M_{22} = M_{33} = M_{44} &= \psi_0(\psi_0^2 - \mu_{13}^2 - \mu_{14}^2) \\ = \psi_0 A, \qquad A &= \psi_0^2 - \mu_{13}^2 - \mu_{14}^2 \end{aligned} \tag{3.7-12}$$

$$M_{12} = M_{34} = 0$$

$$M_{13} = M_{24} = -\mu_{13} A$$

$$M_{14} = -M_{23} = -\mu_{14} A$$

$$|M| = A^2$$

The probability density of the four random variables is therefore

$$\frac{1}{4\pi^2 A} \exp - \frac{1}{2A} [\psi_0(I_1^2 + I_2^2 + I_3^2 + I_4^2)$$

$$- 2\mu_{13}(I_1 I_3 + I_2 I_4) - 2\mu_{14}(I_1 I_4 - I_2 I_3)]$$

where we have written I_1, I_2, I_3, I_4 for $I_{c1}, I_{s1}, I_{c2}, I_{s2}$. We now make the transformation

$$I_1 = R_1 \cos\theta_1 \qquad I_3 = R_2 \cos\theta_2$$

$$I_2 = R_1 \sin\theta_1 \qquad I_4 = R_2 \sin\theta_2$$

and average the resulting probability density over θ_1 and θ_2 in order to get the probability that R_1 and R_2 lie in dR_1 and dR_2. It is

$$\frac{R_1 R_2\, dR_1\, dR_2}{4\pi^2 A} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \exp$$

$$-\frac{1}{2A}[\psi_0 R_1^2 + \psi_0 R_2^2 - 2\mu_{13} R_1 R_2 \cos(\theta_2 - \theta_1) - 2\mu_{14} R_1 R_2 \sin(\theta_2 - \theta_1)]$$

Since the integrand is a periodic function of θ_2 we may integrate from $\theta_2 = \theta_1$ to $\theta_2 = \theta_1 + 2\pi$ instead of from 0 to 2π. This integration gives the Bessel function, I_0, of the first kind with imaginary argument. The resulting probability density for R_1 and R_2 is

$$\frac{R_1 R_2}{A} I_0\left(\frac{R_1 R_2}{A}[\mu_{13}^2 + \mu_{14}^2]^{1/2}\right) \exp -\frac{\psi_0}{2A}(R_1^2 + R_2^2) \qquad (3.7\text{–}13)$$

where, from (3.7–12),

$$A = \psi_0^2 - \mu_{13}^2 - \mu_{14}^2$$

μ_{13} and μ_{14} are given by (3.7–11). Of course, R_1 and R_2 are always positive.

For an ideal band pass filter with cut-offs at f_a and f_b we set

$$f_m = \frac{f_b + f_a}{2}, \qquad w(f) = w_0 \quad \text{for} \quad f_a < f < f_b$$

and obtain

$$\psi_0 = w_0(f_b - f_a)$$

$$\mu_{13} = \int_{f_a}^{f_b} w_0 \cos 2\pi(f - f_m)\tau\, df = \frac{w_0 \sin \pi(f_b - f_a)\tau}{\pi\tau}$$

$$\mu_{14} = \int_{f_a}^{f_b} w_0 \sin 2\pi(f - f_m)\tau\, df = 0$$

The I_0 term in (3.7–13), which furnishes the correlation between R_1 and R_2, becomes

$$I_0\left(\frac{R_1 R_2}{\psi_0} \frac{\dfrac{\sin x}{x}}{1 - \dfrac{\sin^2 x}{x^2}}\right)$$

where x is $\pi(f_b - f_a)\tau$. When x is a multiple of π, R_1 and R_2 are independent random variables. When x is zero R_1 and R_2 are equal. Hence we may say, roughly, that the period of fluctuation of R is the time it takes x to increase from 0 to π or $(f_b - f_a)^{-1}$. This is related to the result given in the next section, namely that the expected number of maxima of the envelope is $.641\ (f_b - f_a)$ per second.

3.8 Maxima of R

Here we wish to study the distribution of the maxima of R.* Our work is based upon the expression, cf. (3.6-1),

$$-dR\,dt \int_{-\infty}^{0} p(R, 0, R'')R''\,dR'' \tag{3.8-1}$$

for the probability that a maximum of R falls within the elementary rectangle $dR\,dt$. $p(R, R', R'')$ is the probability density for the three dimensional distribution of R, R', R'' where the primes denote differentiation with respect to t.

We shall determine $p(R, R', R'')$ from the probability density of I_c, I_s', I_c'', I_s, I_c', I_s'', which we shall denote by $x_1, x_2, \cdots x_6$. The interchange of I_s' and I_c' is suggested by the later work. It is convenient to introduce the notation

$$\begin{aligned} b_n &= (2\pi)^n \int_0^\infty w(f)(f - f_m)^n\,df \\ b_0 &= \psi_0 \end{aligned} \tag{3.8-2}$$

where f_m is the mid-band frequency, i.e., the frequency chosen in the definition of the envelope R. b_n is seen to be analogous to the derivatives of $\psi(\tau)$ at $\tau = 0$.

From the definitions (3.7-3) of I_c and I_s we obtain the second moments

$$\begin{aligned} \overline{x_1^2} &= \overline{I_c^2} = \psi_0 = b_0 \\ \overline{x_4^2} &= \overline{I_s^2} = b_0 \\ \overline{x_2^2} &= \overline{I_s'^2} = \sum_1^N w(f_n)\Delta f 4\pi^2(f_n - f_m)^2 = b_2 \\ \overline{x_5^2} &= \overline{I_c'^2} = b_2 \\ \overline{x_3^2} &= \overline{I_c''^2} = b_4 \\ \overline{x_6^2} &= \overline{I_s''^2} = b_4 \end{aligned}$$

* Incidentally, most of the analysis of this section was originally developed in a study of the stability of repeaters in a loaded telephone transmission line. The envelope, R, was associated with the "returned current" produced by reflections from line irregularities. However, the study fell short of its object and the only results which seemed worth salvaging at the time were given in reference[25] cited in Section 3.3.

$$
\begin{aligned}
\overline{x_1 x_2} &= \overline{I_c I_s'} = \sum_1^N w(f_n)\Delta f 2\pi(f_n - f_m) = b_1 \\
\overline{x_4 x_5} &= \overline{I_s I_c'} = -b_1 \\
\overline{x_1 x_3} &= \overline{I_c I_c''} = -\sum_1^N w(f)\Delta f 4\pi^2 (f_n - f_m)^2 = -b_2 \\
\overline{x_4 x_6} &= \overline{I_s I_s''} = -b_2 \\
\overline{x_2 x_3} &= \overline{I_s' I_c''} = -b_3 \\
\overline{x_5 x_6} &= \overline{I_c' I_s''} = b_3
\end{aligned}
$$

All of the other second moments are zero. The moment matrix M is thus

$$
M = \begin{bmatrix}
b_0 & b_1 & -b_2 & 0 & 0 & 0 \\
b_1 & b_2 & -b_3 & 0 & 0 & 0 \\
-b_2 & -b_3 & b_4 & 0 & 0 & 0 \\
0 & 0 & 0 & b_0 & -b_1 & -b_2 \\
0 & 0 & 0 & -b_1 & b_2 & b_3 \\
0 & 0 & 0 & -b_2 & b_3 & b_4
\end{bmatrix}
$$

The adjoint matrix is

$$
\begin{bmatrix}
B_0 & B_1 & -B_2 & 0 & 0 & 0 \\
B_1 & B_{22} & -B_3 & 0 & 0 & 0 \\
-B_2 & -B_3 & B_4 & 0 & 0 & 0 \\
0 & 0 & 0 & B_0 & -B_1 & -B_2 \\
0 & 0 & 0 & -B_1 & B_{22} & B_3 \\
0 & 0 & 0 & -B_2 & B_3 & B_4
\end{bmatrix}
$$

$$
\begin{aligned}
B_0 &= (b_2 b_4 - b_3^2)B & B_{22} &= (b_0 b_4 - b_2^2)B \\
B_1 &= -(b_1 b_4 - b_2 b_3)B & B_3 &= -(b_0 b_3 - b_1 b_2)B \\
B_2 &= (b_1 b_3 - b_2^2)B & B_4 &= (b_0 b_2 - b_1^2)B
\end{aligned}
\qquad (3.8\text{–}3)
$$

$$
\begin{aligned}
B &= b_0 b_2 b_4 + 2\, b_1 b_2 b_3 \\
&\quad - b_2^3 - b_0 b_3^2 - b_4 b_1^2 \\
|M| &= B^2
\end{aligned}
$$

where B is the determinant of the third order matrices in the upper left and lower right corners of M.

As in the earlier work, the distribution of $x_1, \cdots, x_6$ is normal in six dimensions. The exponent is $-[2\,|M|\,]^{-1}$ times

$$
\begin{aligned}
B_0(x_1^2 + x_4^2) + 2B_1(x_1 x_2 - x_4 x_5) &- 2B_2(x_1 x_3 + x_4 x_6) \\
+ B_{22}(x_2^2 + x_5^2) &- 2B_3(x_2 x_3 - x_5 x_6) \qquad (3.8\text{–}4) \\
&+ B_4(x_3^2 + x_6^2)
\end{aligned}
$$

In line with the earlier work we set

$$
\begin{aligned}
x_1 = I_c = R\cos\theta \qquad x_4 &= I_s = R\sin\theta \\
x_2 = I_s' &= R'\sin\theta + R\cos\theta\theta' \\
x_5 = I_c' &= R'\cos\theta - R\sin\theta\theta' \\
x_3 = I_c'' &= R''\cos\theta - 2R'\sin\theta\theta' \\
&\quad - R\cos\theta\theta'^2 - R\sin\theta\theta'' \\
x_6 = I_s'' &= R''\sin\theta + 2R'\cos\theta\theta' \\
&\quad - R\sin\theta\theta'^2 + R\cos\theta\theta''
\end{aligned}
$$

The angle θ varies from 0 to 2π and θ' and θ'' vary from $-\infty$ to $+\infty$. By forming the Jacobian it may be shown that

$$dx_1\, dx_2 \cdots dx_6 = R^3\, dR\, dR'\, dR''\, d\theta\, d\theta'\, d\theta''$$

Also, the quantities in (3.8–4) are

$$
\begin{aligned}
x_1^2 + x_4^2 &= R^2 \qquad\qquad x_1x_3 + x_4x_6 = RR'' - R^2\theta'^2 \\
x_1x_2 - x_4x_5 &= R^2\theta' \qquad x_2^2 + x_5^2 = R'^2 + R^2\theta'^2 \\
x_2x_3 - x_5x_6 &= RR''\theta' - 2R'^2\theta' - R'R\theta'' - R^2\theta'^3 \\
x_3^2 + x_6^2 &= R''^2 - 2RR''\theta'^2 + 4R'^2\theta'^2 + 4RR'\theta'\theta'' \\
&\quad + R^2\theta'^4 + R^2\theta''^2
\end{aligned}
$$

The expression for $p(R, 0, R'')$ is obtained when we set these values of the x's in (3.8–4) and integrate the resulting probability density over the ranges of θ, θ', θ'':

$$
\begin{aligned}
p(R, 0, R'') = \frac{R^3}{8\pi^3 B} \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\theta' \int_{-\infty}^{+\infty} d\theta'' \qquad (3.8\text{–}5) \\
\exp -\frac{1}{2B^2}\,[B_0R^2 + 2B_1R^2\theta' - 2B_2(RR'' - R^2\theta'^2) \\
+ B_{22}R^2\theta'^2 - 2B_3R\theta'(R'' - R\theta'^2) \\
+ B_4(R''^2 - 2RR''\theta'^2 + R^2\theta'^4 + R^2\theta''^2)]
\end{aligned}
$$

The integrations with respect to θ and θ'' may be performed at once leaving $p(R, 0, R'')$ expressed as a single integral which, unfortunately, appears to be difficult to handle. For this reason we assume that $w(f)$ is symmetrical about the mid-band frequency f_m. From (3.8–2), b_1 and b_3 are zero and from (3.8–3), B_1 and B_3 are zero.

With this assumption (3.8-5) yields

$$p(R, 0, R'') = R^2(2\pi)^{-3/2} B_4^{-1/2} \int_{-\infty}^{+\infty} d\theta' \tag{3.8-6}$$

$$\exp -\frac{1}{2B^2}[B_0R^2 + R([B_{22} + 2B_2]R\theta'^2 - 2B_2R'') + B_4(R'' - R\theta'^2)^2]$$

The probability that a maximum occurs in the elementary rectangle $dR\, dt$ is, from (3.8-1), $p(t, R)\, dR\, dt$ where

$$p(t, R) = -\int_{-\infty}^{0} p(R, 0, R'')R''\, dR'' \tag{3.8-7}$$

We put (3.8-6) in this expression and make the following change of variables.

$$x = \frac{B_4^{1/2}}{\sqrt{2}\, B} R\theta'^2, \qquad y = -\frac{B_4^{1/2}}{\sqrt{2}\, B} R''$$

$$z = -\frac{B_2}{\sqrt{2B_4}\, B} R = \frac{b_2^2}{\sqrt{2B_4}} R \tag{3.8-8}$$

$$b = -\frac{(B_{22} + 2B_2)}{2B\, b_2^2} = \left[\frac{3}{2} - \frac{b_0 b_4}{2b_2^2}\right] = \tfrac{1}{2}(3 - a^2)$$

$$a^2 = \frac{B_0}{2B^2}\frac{2B_4}{b_2^4} = \frac{b_0 b_4}{b_2^2}$$

where we have used the expressions for the B's obtained by setting b_1 and b_3 to zero in (3.8-3). Thus

$$p(t, R) = \frac{4}{b_0 b_2^4}\left(\frac{Bz}{2\pi}\right)^{3/2} \int_0^{\infty} y\, dy \int_0^{\infty} x^{-1/2}\, dx \tag{3.8-9}$$

$$\exp\left[-a^2z^2 + 2bzx + 2zy - (x + y)^2\right]$$

As was to be expected, this expression shows that $p(t, R)$ is independent of t.

A series for $p(t, R)$ may be obtained by expanding $\exp 2z(y + bx)$ and then integrating termwise. We use

$$\int_0^{\infty} dy \int_0^{\infty} dx\, x^{\mu} y^{\gamma} e^{-(x+y)^2} = \frac{\sqrt{\pi}}{2^{\mu+\gamma+2}} \frac{\Gamma(\gamma + 1)\Gamma(\mu + 1)}{\Gamma\left(\dfrac{\mu + \gamma + 3}{2}\right)}$$

which may be evaluated by setting

$$x = \rho^2 \cos^2 \varphi, \qquad y = \rho^2 \sin^2 \varphi$$

The double integral in (3.8-9) becomes

$$e^{-a^2z^2}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(2z)^n}{n!}\sum_{m=0}^{n}\frac{n!\,b^m}{m!\,(n-m)!}\frac{\Gamma(m+\frac{1}{2})\Gamma(n-m+2)}{2^{n+2}\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}$$

$$= \pi 2^{-5/2}\sum_{n=0}^{\infty}\frac{z^n e^{-a^2z^2}}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}A_n$$

where $A_0 = 1$ and

$$A_n = \sum_{m=0}^{n}\frac{(\frac{1}{2})(\frac{3}{2})\cdots(m-\frac{1}{2})}{m!}(n-m+1)b^m, \qquad 0 < n \qquad (3.8\text{-}10)$$

$$A_n \sim (n+1)(1-b)^{-1/2} - \frac{b}{2}(1-b)^{-3/2}, \qquad n \text{ large}$$

The term corresponding to $m = 0$ in (3.8-10) is $n + 1$.

We thus obtain

$$\begin{aligned} p(t, R) &= \frac{e^{-a^2z^2}}{4b_0b_2^{\frac{1}{2}}}\frac{(Bz)^{3/2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}A_n \\ &= \frac{e^{-a^2z^2}}{4\sqrt{\pi}}\frac{b_2^{1/2}}{b_0}(a^2-1)^{3/2}z^{3/2}\sum_{n=0}^{\infty}\frac{z^n A_n}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)} \end{aligned} \qquad (3.8\text{-}11)$$

We are interested in the expected number, N, of maxima per second. From the similar work for I, it follows that N is the coefficient of dt when (3.8-1) is integrated with respect to R from 0 to ∞. Thus from (3.8-7) and

$$\begin{aligned} dR &= \sqrt{2B_4}\,b_2^{-2}\,dz = (2b_0B)^{1/2}b_2^{-3/2}\,dz \\ &= [2b_0(a^2-1)]^{1/2}\,dz \end{aligned}$$

we find

$$\begin{aligned} N &= \int_0^{\infty} p(t, R)\,dR \\ &= \frac{(a^2-1)^2}{(2a)^{5/2}}\left(\frac{b_2}{\pi b_0}\right)^{1/2}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{5}{4}\right)}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}\frac{A_n}{a^n} \end{aligned} \qquad (3.8\text{-}12)$$

Equations (3.8-11) and (3.8-12) have been derived on the assumption that $w(f)$ is symmetrical about f_m, i.e. the band pass filter attenuation is symmetrical about the mid-band frequency. We now go a step further and assume an ideal band pass filter:

$$
\begin{aligned}
w(f) &= w_0 \qquad f_a < f < f_b \\
w(f) &= 0 \qquad \text{otherwise} \qquad\qquad (3.8\text{-}13) \\
2f_m &= f_a + f_b
\end{aligned}
$$

Putting these in (3.8–2) we obtain *zero* for b_1 and b_3 and also

$$
\begin{aligned}
b_0 &= w_0(f_b - f_a) = \psi_0 \\
b_2 &= \frac{\pi^2 w_0}{3}(f_b - f_a)^3 \\
b_4 &= \frac{\pi^4 w_0}{5}(f_b - f_a)^5 \\
a^2 &= \tfrac{9}{5} \qquad\qquad (3.8\text{-}14) \\
b &= \tfrac{1}{2}(3 - a^2) = \tfrac{3}{5} \\
R &= [2b_0(a^2 - 1)]^{1/2} z = [\tfrac{8}{5}\psi_0]^{1/2} z \\
\left(\frac{b_2}{\pi b_0}\right)^{1/2} &= \left[\frac{\pi}{3}\right]^{1/2}(f_b - f_a), \qquad a^2 z^2 = \frac{9R^2}{8\psi_0}
\end{aligned}
$$

n	A_n	n	A_n
0	1	4	6.775
1	2.3	5	8.333
2	3.735	6	9.9002
3	5.238	7	11.4736

$$A_n \sim 1.5811\, n + .3953$$

From (3.8–12) we find that the expected number of maxima per second of the envelope is

$$N = .64110\,(f_b - f_a) \qquad (3.8\text{-}15)$$

assuming an ideal band pass filter.

The distribution of the maxima of R for an ideal band pass filter may be obtained by placing the results of (3.8–14) in (3.8–11). This gives

$$
p(t, R)\, dR = \frac{dR}{\psi_0^{1/2}} \frac{(f_b - f_a)}{4} \sqrt{\frac{\pi}{3}} \left(\frac{4z}{5}\right)^{3/2} e^{-a^2 z^2} \sum_{n=0}^{\infty} \frac{z^n A_n}{\Gamma\left(\frac{n}{2} + \frac{7}{4}\right)}
$$

It is convenient to define y as the ratio

$$y = \frac{R}{\text{r.m.s. } I(t)} = \frac{R}{\psi_0^{1/2}} = \left(\tfrac{8}{3}\right)^{1/2} z$$

where R is understood to correspond to a maximum of the envelope. Since the value of R corresponding to a maximum of the envelope selected at random is a random variable, y is also a random variable. Its probability density is $p_R(y)$, where

$$p_R(y)\, dy = \frac{p(t, R)\, dR}{0.64110(f_b - f_a)}$$

$p_R(y)$ has been computed and is plotted as a function of y in Fig. 3.

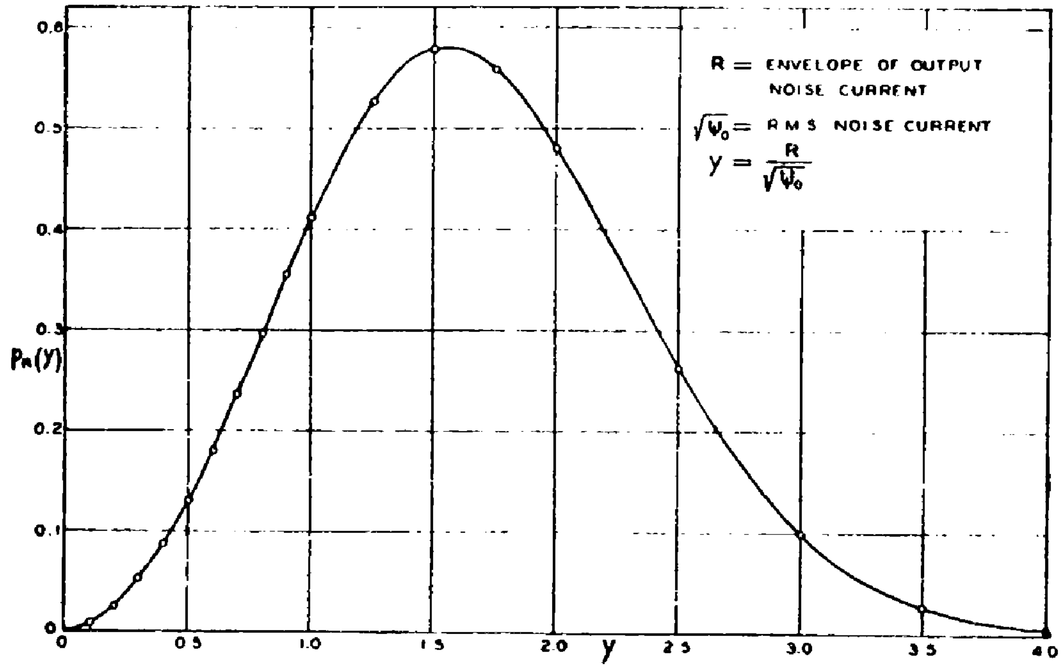

Fig. 3—Distribution of maxima of envelope of noise current. Noise through ideal band-pass filter.

$\frac{p_R(y)}{\sqrt{\psi_0}}\, dR$ = probability that a maximum of R selected at random lies between R and $R + dR$.

The distribution function $P(R_{max} < y\sqrt{\psi_0})$ defined by

$$P(R_{max} < y\sqrt{\psi_0}) = \int_0^y p_R(y)\, dy$$

and which gives the probability that a maximum of the envelope selected at random is less than a specified value $y\sqrt{\psi_0} = R$, is plotted in Fig. 4 together with other curves of the same nature.

When y is large, say greater than 2.5,

$$p_R(y) \sim \frac{\sqrt{\frac{\pi}{6}}}{.64110}(y^2 - 1)e^{-y^2/2}$$

$$P(R_{max} < y\sqrt{\psi_0}) \sim 1 - \frac{\sqrt{\frac{\pi}{6}}}{.64110} y e^{-y^2/2}$$

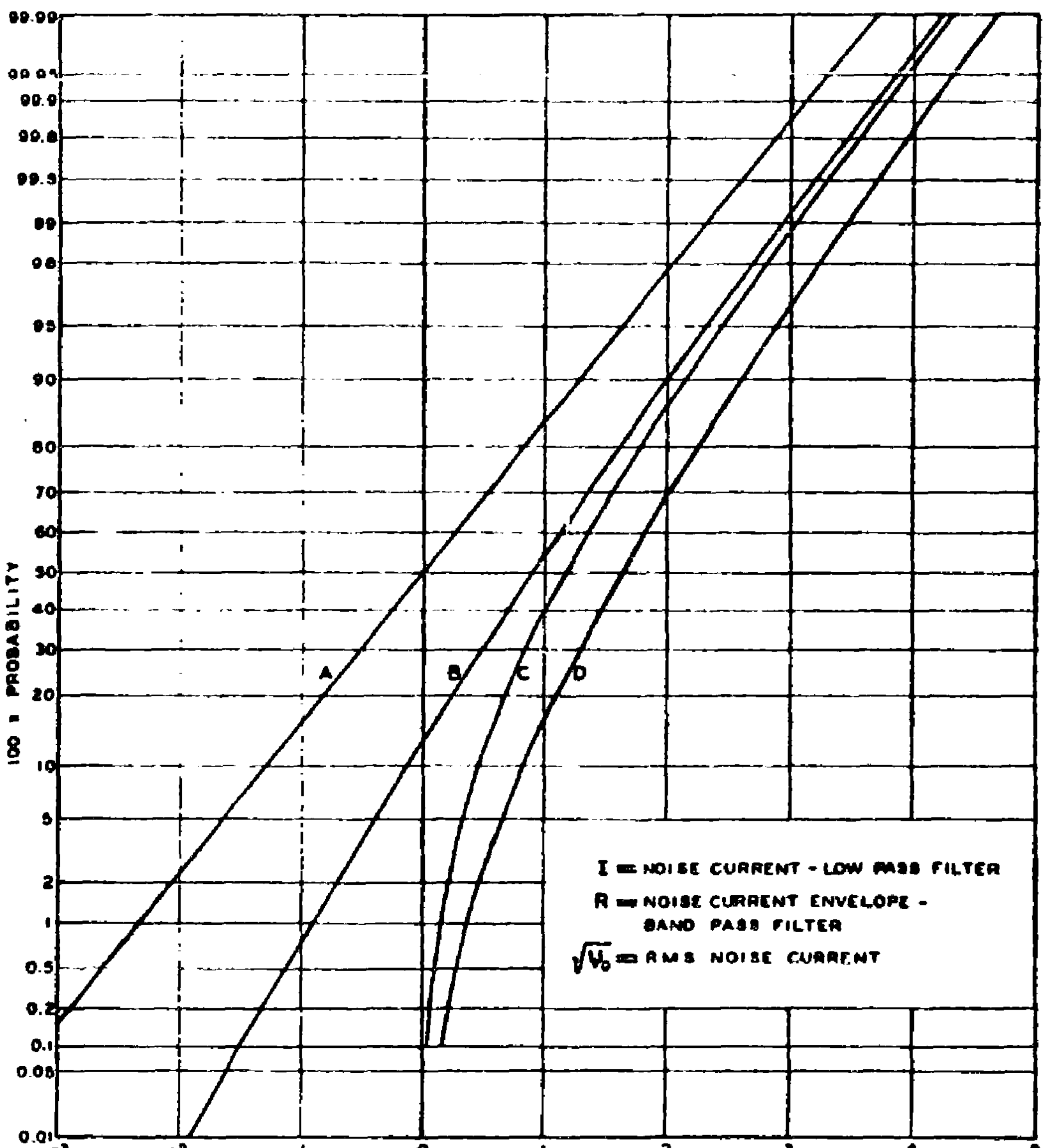

Fig. 4—Distribution of maxima

$A = P(I < y\sqrt{\psi_0})$ = probability of I being less than $y\sqrt{\psi_0}$. Similarly $C = P(R < y\sqrt{\psi_0})$.

$B = P(I \max < y\sqrt{\psi_0})$ = probability of random maximum of I being less than $y\sqrt{\psi_0}$. Similarly $D = P(R \max < y\sqrt{\psi_0})$.

The asymptotic expression for $p_R(y)$ may be obtained from the integral (3.8-9) for $p(t, R)$. Indeed, replacing the variables of integration x, y in (3.8-9) by

$$x' = x$$

$$y' = x + y,$$

integrating a portion of the y' integral by parts, and assuming $b < 1$ ($a^2 \geq 1$, by Schwarz's inequality, so that $b \leq 1$ always) leads to

$$p(t, R) \sim \left(\frac{b_2}{2\pi}\right)^{\frac{1}{2}} \frac{e^{-R^2/2\psi_0}}{\psi_0} \left(\frac{R^2}{\psi_0} - 1\right)$$

when R is large.

If, instead of an ideal band pass filter, we assume that $w(f)$ is given by

$$w(f) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(f-f_m)^2/2\sigma^2}, \qquad f_m \gg \sigma \tag{3.8-16}$$

we find that

$$b_0 = 1$$

$$b_2 = 4\pi^2\sigma^2$$

$$b_4 = 16\pi^4 \cdot 3\sigma^4$$

$$a^2 = 3, b = 0$$

$$A_n = (n + 1)$$

Some rough work indicates that the sum of the series in (3.8-12) is near 3.97. This gives the expected number of maxima of the envelope as

$$N = 2.52\sigma \tag{3.8-17}$$

per second.

The pass band is determined by σ. It appears difficult to compare this with an ideal band pass filter. If we use the fact that the filter given by

$$w(f) = w_0 \exp\left[-\pi\left(\frac{f - f_m}{f_b - f_a}\right)^2\right]$$

passes the same average amount of power as does an ideal band pass filter whose pass band is $f_b - f_a$, we have

$$f_b - f_a = \sigma\sqrt{2\pi}$$

and the expression for N becomes $1.006\,(f_b - f_a)$.

3.9 Energy Fluctuation

Some information regarding the statistical behavior of the random variable

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt \tag{3.9-1}$$

where $I(t)$ is a noise current and t_1 is chosen at random, has been given in a recent article.[85] Here we study this behavior from a somewhat different point of view.

If we agree to use the representations (2.8-1) or (2.8-6) we may write, as in the paper, the random variable E as

$$E = \int_{-T/2}^{T/2} I^2(t)\, dt \tag{3.9-2}$$

where the randomness on the right is due either to the a_n's and b_n's if (2.8-1) is used or to the φ_n's if (2.8-6) is used.

The average value of E is m_T where, from (3.1-2),

$$\begin{aligned} \bar{E} = m_T &= \int_{-T/2}^{T/2} \overline{I^2(t)}\, dt = \int_{-T/2}^{T/2} \psi(0)\, dt = T\psi_0 \\ &= T \int_0^\infty w(f)\, df \end{aligned} \tag{3.9-3}$$

The second moment of E is

$$\overline{E^2} = \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\, \overline{I^2(t_1) I^2(t_2)} \tag{3.9-4}$$

If, for the time being, we set t_2 equal to $t_1 + \tau$, it is seen from section 3.2 that we have an expression for the probability density of $I(t_1)$ and $I(t_1 + \tau)$ and hence we may obtain the required average:

$$\overline{I_1^2 I_2^2} = \frac{1}{2\pi A} \int_{-\infty}^{+\infty} dI_1 \int_{-\infty}^{+\infty} dI_2\, I_1^2 I_2^2 \exp \left(-\frac{1}{2A^2} (\psi_0 I_1^2 + \psi_0 I_2^2 - 2\psi_\tau I_1 I_2) \right) \tag{3.9-5}$$

$$A^2 = \psi_0^2 - \psi_\tau^2, \quad I_1 = I(t_1), \quad I_2 = I(t_1 + \tau) = I(t_2)$$

The integral may be evaluated by (3.5-6) when we set

$$\begin{aligned} I_1 = Ax \sqrt{\frac{2}{\psi_0}}, &\qquad I_2 = Ay \sqrt{\frac{2}{\psi_0}} \\ \psi_\tau &= -\psi_0 \cos \varphi \\ A &= \psi_0 \sin \varphi \end{aligned} \tag{3.9-6}$$

[85] "Filtered Thermal Noise—Fluctuation of Energy as a Function of Interval Length", *Jour. Acous. Soc. Am.*, 14 (1943), 216-227.

Thus

$$
\begin{aligned}
\overline{I_1^2 I_2^2} &= \psi_0^2(1 + 2\cos^2\varphi) \\
&= \psi_0^2 + 2\psi_\tau^2
\end{aligned}
\tag{3.9-7}
$$

Incidentally, this gives an expression for the correlation function of $I^2(t)$. Replacing τ by its value of $t_2 - t_1$ and returning to (3.9-4),

$$
\overline{E^2} = T^2\psi_0^2 + 2\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \psi^2(t_2 - t_1) \tag{3.9-8}
$$

When we introduce σ_T, the standard deviation of E, and use

$$
\sigma_T^2 = \overline{E^2} - m_T^2
$$

we obtain

$$
\begin{aligned}
\sigma_T^2 = \overline{(E - \bar{E})^2} &= 2\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \psi^2(t_2 - t_1) \\
&= 4\int_0^T (T - x)\psi^2(x)\, dx
\end{aligned}
$$

where the second line may be obtained from the first either by changing the variables of integration, as in (3.9-27), or by the method used below in dealing with $\overline{E^3}$. I am indebted to Prof. Kac for pointing out the advantage obtained by reducing the double integral to a single integral. It should be noted that the limits of integration $-T/2$, $T/2$ in the double integral may be replaced by 0, T by making the change of variable $t = t' - T/2$ for both t_1 and t_2.

When we use

$$
\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \, df \tag{2.1-6}
$$

we obtain the result stated in the paper, namely,

$$
\sigma_T^2 = \int_0^\infty w(f_1)\, df_1 \int_0^\infty w(f_2)\, df_2 \left[\frac{\sin^2 \pi(f_1 + f_2)T}{\pi^2(f_1 + f_2)^2} + \frac{\sin^2 \pi(f_1 - f_2)T}{\pi^2(f_1 - f_2)^2}\right] \tag{3.9-9}
$$

If this formula is applied to a relatively narrow band-pass filter and if $T(f_b - f_a) >> 1$ the contribution of the $f_1 + f_2$ term may be neglected and we have the approximation

$$
\begin{aligned}
\sigma_T^2 &= \int_{f_a}^{f_b} w_0\, df_1 \int_{-\infty}^{+\infty} w_0\, df_2 \frac{\sin^2 \pi(f_1 - f_2)T}{\pi^2(f_1 - f_2)^2} \\
&= w_0^2 T(f_b - f_a) \\
&= w_0 m_T
\end{aligned}
\tag{3.9-10}
$$

where, from (3.9–3)

$$m_T = w_0 T(f_b - f_a) \tag{3.9–11}$$

The third moment E^3 may be computed in the same way. However, in this case it pays to introduce the characteristic function for the distribution of $I(t_1)$, $I(t_2)$, $I(t_3)$. Since this distribution is normal its characteristic function is

$$\begin{aligned} \text{Average } \exp\ [iz_1 I_1 + iz_2 I_2 + iz_3 I_3] \\ = \exp -\Big[\frac{\psi_0}{2}(z_1^2 + z_2^2 + z_3^2) + \psi(t_2 - t_1)z_1 z_2 \\ + \psi(t_3 - t_1)z_1 z_3 + \psi(t_3 - t_2)z_2 z_3\Big] \end{aligned} \tag{3.9–12}$$

From the definition of the characteristic function it follows that

$$\begin{aligned} \overline{I_1^2 I_2^2 I_3^2} &= -\text{coeff. of } \frac{z_1^2 z_2^2 z_3^2}{2!2!2!} \text{ in ch. f.} \\ &= \psi_0^3 + 2\psi_0(\psi_{21}^2 + \psi_{31}^2 + \psi_{32}^2) \\ &\quad + 8\psi_{21}\psi_{31}\psi_{32} \end{aligned} \tag{3.9–13}$$

where we have written ψ_{21} for $\psi(t_2 - t_1)$, etc. When (3.9–13) is multiplied by $dt_1\, dt_2\, dt_3$, the variables integrated from 0 to T, and the above double integral expression for σ_T^2 used, we find

$$\overline{(E - \bar{E})^3} = 2!2^2 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \psi_{21}\psi_{31}\psi_{32}\,.$$

Denoting the triple integral on the right by J and differentiating,

$$\begin{aligned} \frac{dJ}{dT} &= 3\int_0^T dt_1 \int_0^T dt_2 \psi(t_2 - t_1)\psi(T - t_1)\psi(T - t_2) \\ &= 3\int_0^T dx \int_0^T dy \psi(x - y)\psi(x)\psi(y) \\ &= 6\int_0^T dx \int_0^x dy \psi(x - y)\psi(x)\psi(y) \end{aligned}$$

In going from the first line to the second t_1 and t_2 were replaced by $T - x$ and $T - y$, respectively. In going from the second to the third use was made of the relations symbolized by

$$\begin{aligned} \int_0^T dx \int_0^T dy &= \int_0^T dx \int_0^x dy + \int_0^T dx \int_x^T dy \\ &= \int_0^T dx \int_0^x dy + \int_0^T dy \int_0^y dx \end{aligned}$$

and of the fact that the integrand is symmetrical in x and y. Integrating dJ/dT with respect to T from 0 to T_1, using the formula

$$\int_0^{T_1} dT \int_0^T f(x)\,dx = \int_0^{T_1} (T_1 - x)f(x)\,dx,$$

noting that J is zero when T is zero, and dropping the subscript on T_1 finally gives

$$\overline{(E - \bar{E})^3} = 48 \int_0^T dx \int_0^x dy (T - x)\psi(x)\psi(y)\psi(x - y).$$

$\overline{E^4}$ may be treated in a similar way. It is found that

$$\overline{(E - \bar{E})^4} - 3\overline{(E - \bar{E})^2}^2 = 3!2^3 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \int_0^T dt_4 \psi_{21}\psi_{31}\psi_{42}\psi_{43}$$

which may be reduced to the sum of two triple integrals. It is interesting to note that the expression on the left is the fourth semi-invariant of the random variable E and gives us a measure of the peakedness of the distribution (kurtosis). Likewise, the second and third moments about the mean are the second and third semi-invariants of E. This suggests that possibly the higher semi-invariants may also be expressed as similar multiple integrals.

So far, in this section, we have been speaking of the statistical constants of E. The determination of an exact expression for the probability density of E, in which T occurs as a parameter, seems to be quite difficult.

When T is very small E is approximately $I^2(t)T$. The probability that E lies in dE is the probability that the current lies in $-I$, $-I - dI$ plus the probability that the current lies in I, $I + dI$:

$$\frac{2dI}{\sqrt{2\pi\psi_0}} \exp - \frac{I^2}{2\psi_0} = (2\pi\psi_0 ET)^{-1/2} \exp - \frac{E}{2\psi_0 T} dE \quad (3.9\text{–}14)$$

where E is positive,

$$I = \left(\frac{E}{T}\right)^{1/2}, \qquad dI = \frac{1}{2}(ET)^{-1/2}\,dE$$

and T is assumed to be so small that $I(t)$ does not change appreciably during an interval of length T.

When T is very large we may divide it into a number of intervals, say n, each of length T/n. Let E_r be the contribution of the r th interval. The energy E for the entire interval is then

$$E = E_1 + E_2 + \cdots + E_n$$

If the sub-intervals are large enough the E_r's are substantially independent random variables. If in addition n is large enough E is distributed normally, approximately. Hence when T is very large the probability that E lies in dE is

$$\frac{dE}{\sigma_T\sqrt{2\pi}} \exp - \frac{(E - m_T)^2}{2\sigma_T^2} \tag{3.9–15}$$

where

$$\begin{aligned} m_T &= T \int_0^\infty w(f)\, df \\ \sigma_T^2 &= T \int_0^\infty w^2(f)\, df \end{aligned} \tag{3.9–16}$$

the second relation being obtained by letting $T \to \infty$ in (3.9–9). The analogy with Campbell's theorem, section 1.2, is evident. When we deal with a band pass filter we may use (3.9–10) and (3.9–11).

Consider a relatively narrow band pass filter such that we may find a T for which $Tf_a >> 2\pi$ but $T(f_b - f_a) << .64$. Thus several cycles of frequency f_a are contained in T but, from (3.8–15), the envelope does not change appreciably during this interval. Thus throughout this interval $I(t)$ may be considered to be a sine wave of amplitude R. The corresponding value of E is approximately

$$E = T\frac{R^2}{2}$$

where the distribution of the envelope R is given by (3.7–10). From this it follows that the probability of E lying in dE is

$$\frac{dE}{\psi_0 T} \exp - \frac{E}{\psi_0 T} = \frac{dE}{m_T} e^{-E/m_T} \tag{3.9–17}$$

when E is small but not too small.

When we look at (3.9–14) and (3.9–17) we observe that they are of the form

$$\frac{a^{n+1}E^n}{\Gamma(n+1)} e^{-aE}\, dE \tag{3.9–18}$$

Moreover, the normal law (3.9–15), may be obtained from this by letting n become large. This suggests that an approximate expression for the distribution of E is given by (3.9–18) when a and n are selected so as to give the values of m_T and σ_T obtained from (3.9–3) and (3.9–9). This gives

$$a = \frac{m_T}{\sigma_T^2}, \qquad n + 1 = \frac{m_T^2}{\sigma_T^2} \tag{3.9–19}$$

and if we drop the subscript T and substitute the value of a in (3.9-18) we get

$$\frac{\left(\frac{mE}{\sigma^2}\right)^n}{\Gamma(n+1)} \exp\left(-\frac{mE}{\sigma^2}\right) d\left(\frac{mE}{\sigma^2}\right), \qquad n = \frac{m^2}{\sigma^2} - 1 \qquad (3.9\text{–}20)$$

An idea of how this distribution behaves may be obtained from the following table:

n	$T(f_b - f_a)$	$x_{.25}$	$x_{.50}$	$x_{.75}$	$\frac{x_{.25}}{x_{.50}}$	$\frac{x_{.75}}{x_{.50}}$
0	0	.29	.695	1.39	.415	2.00
1	1.45	.96	1.68	2.69	.572	1.60
2	2.4	1.73	2.67	3.94	.647	1.47
3	3.4	2.54	3.67	5.12	.692	1.39
5	5.4	4.22	5.67	7.42	.744	1.31
10	10.5	8.63	10.67	13.02	.808	1.22
24	25	21.47	24.67	28.17	.870	1.14
48	50	44.1	48.7	53.5	.905	1.10

where n is the exponent in (3.9-20). The column $T(f_b - f_a)$ holds only for a narrow band pass filter and was obtained by reading the curve y_A in Fig. 1 of the above mentioned paper. The figures in this column are not very accurate. The next three columns give the points which divide the distribution into four intervals of equal probability:

$$x_{.25} = \frac{mE_{.25}}{\sigma^2}, \quad E_{.25} = \text{energy exceeded } 75\% \text{ of time}$$

$$x_{.50} = \frac{mE_{.50}}{\sigma^2}, \quad E_{.50} = \text{energy exceeded } 50\% \text{ of time}$$

$$x_{.75} = \frac{mE_{.75}}{\sigma^2}, \quad E_{.75} = \text{energy exceeded } 25\% \text{ of time}$$

The values in these columns were obtained from Pearson's table of the incomplete gamma function. The last two columns show how the distribution clusters around the average value as the normal law is approached.

For the larger values of n we expected the normal law (3.9-15) to be approached. Since, for this law the 25, 50, and 75 per cent points are at $m - .675\sigma$, m, and $m + .675\sigma$ we have to a first approximation

$$x_{.50} = \frac{m^2}{\sigma^2} = (n+1) = T(f_b - f_a)$$

$$x_{.25} = \frac{m}{\sigma^2}(m - .675\sigma) = x_{.50} - .675\sqrt{x_{.50}} \qquad (3.9\text{–}21)$$

$$x_{.75} = x_{.50} + .675\sqrt{x_{.50}}$$

This agrees with the table.

Thiede[36] has studied the mean square value of the fluctuations of the integral

$$A(t) = \int_{-\infty}^{t} I^2(\tau)e^{-\alpha(t-\tau)}\, d\tau \tag{3.9-22}$$

The reading of a hot wire ammeter through which a current I is passing is proportional to $A(t)$. α is a constant of the meter. Here we study $A(t)$ by

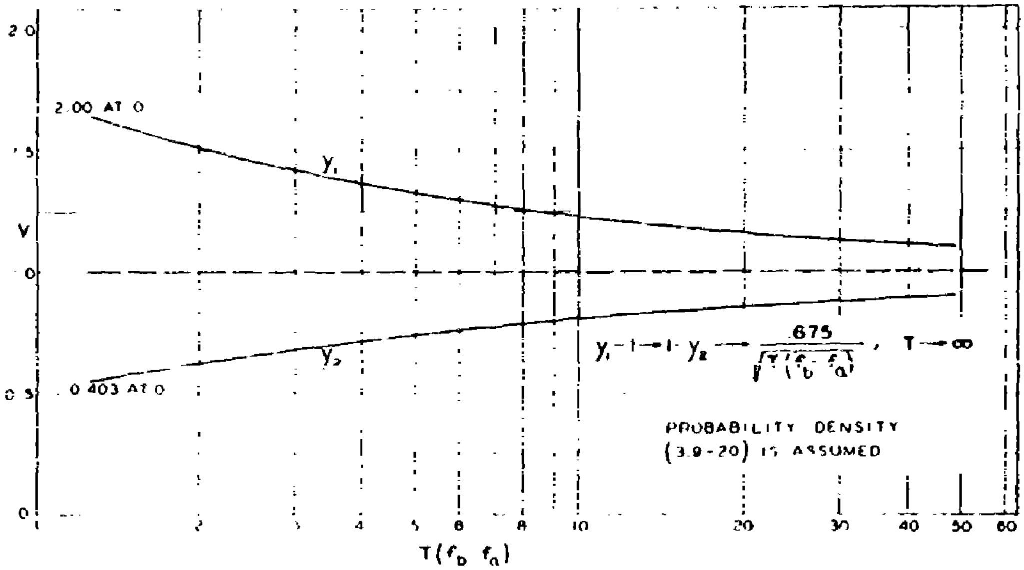

Fig. 5* – Filtered thermal noise—spread of energy fluctuation

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt, \qquad t_1 \text{ random}, \qquad I \text{ is noise current.}$$

$y_1 = E_{.75}/E_{.50}$, $y_2 = E_{.25}/E_{.50}$.
$f_b - f_a$ = band width of filter.

first obtaining its correlation function. This method of approach enables us to extend Thiede's results

The distributed portion of the power spectrum of $A(t)$ is given by (3.9-30). When the power spectrum $w(f)$ of $I(t)$ is zero except over the band $f_a < f < f_b$ where it is w_0, the power spectrum of $A(t)$ is

$$\frac{2w_0^2(f_b - f_a - f)}{\alpha^2 + 4\pi^2 f^2} \quad \text{for} \quad 0 < f < f_b - f_a$$

and is zero from $f_b - f_a$ up to $2f_a$. The spectrum from $2f_a$ to $2f_b$ is not zero, and may be obtained from (3.9-34). The mean square fluctuation of $A(t)$ is given, in the general case, by (3.9-28) and (3.9-32). For the band pass case, when $(f_b - f_a)/\alpha$ is large,

$$\text{r.m.s.}\ \frac{A(t) - \bar{A}}{\bar{A}} = \left[\frac{\alpha}{2(f_b - f_a)}\right]^{1/2}$$

[36] *Elec. Nachr. Tek.*, 13 (1936), 84–95. This is an excellent article.
* Note added in proof. The value of y_2 at 0 should be .415 instead of .403.

We start by setting $\tau = t - u$ which transforms the integral for $A(t)$ into

$$A(t) = \int_0^\infty I^2(t - u)e^{-\alpha u}\, du \tag{3.9–23}$$

In order to obtain the correlation function $\Psi(\tau)$ for $A(t)$ we multiply $A(t)$ by $A(t + \tau)$ and average over all the possible currents

$$\begin{aligned}\Psi(\tau) &= \overline{A(t)A(t + \tau)} \\ &= \int_0^\infty e^{-\alpha u}\, du \int_0^\infty e^{-\alpha v}\, dv \text{ ave. } I^2(t - u)I^2(t + \tau - v)\end{aligned}$$

Just as in (3.9–4) the average in the integrand is the correlation function of $I^2(t)$, the argument being $t + \tau - v - t + u = \tau + u - v$. From (3.9–7) it is seen that this is

$$\psi_0^2 + 2\psi^2(\tau + u - v)$$

where $\psi(\tau)$ is the correlation function of $I(t)$. Hence

$$\Psi(\tau) = \frac{\psi_0^2}{\alpha^2} + 2\int_0^\infty du \int_0^\infty dv\, e^{-\alpha u - \alpha v}\psi^2(\tau + u - v) \tag{3.9–24}$$

From the integral (3.9–23) for $A(t)$ it is seen that the average value of $A(t)$ is

$$\bar{A} = \frac{\overline{I^2}}{\alpha} = \frac{\psi_0}{\alpha} \tag{3.9–25}$$

where we have used

$$\psi_0 = \psi(0) = \int_0^\infty w(f)\, df = \overline{I^2}$$

Using this result again, only this time applying it to $A(t)$, gives

$$\begin{aligned}\overline{A^2(t)} &= \Psi(0) \\ &= \bar{A}^2 + 2\int_0^\infty du \int_0^\infty dv\, e^{-\alpha u - \alpha v}\psi^2(u - v)\end{aligned} \tag{3.9–26}$$

The double integrals may be transformed by means of the change of variable $u + v = x$, $u - v = y$. Then (3.9–24) becomes

$$\begin{aligned}\Psi(\tau) &= \bar{A}^2 + \left[\int_0^\infty dy \int_y^\infty dx + \int_{-\infty}^0 dy \int_{-y}^\infty dx\right] e^{-\alpha x}\psi^2(\tau + y) \\ &= \bar{A}^2 + \frac{1}{\alpha}\int_0^\infty e^{-\alpha y}[\psi^2(\tau + y) + \psi^2(\tau - y)]\, dy\end{aligned} \tag{3.9–27}$$

When we make use of the fact that $\psi(y)$ is an even function of y we see, from (3.9-26), that the mean square fluctuation of $A(t)$ is

$$\overline{(A(t) - \bar{A})^2} = \overline{A^2(t)} - \bar{A}^2 = \frac{2}{\alpha}\int_0^\infty e^{-\alpha y}\psi^2(y)\,dy \qquad (3.9\text{-}28)$$

$\Psi(\tau)$ may be expressed in terms of integrals involving the power spectrum $w(f)$ of $I(t)$. The work starts with (3.9-24) and is much the same as in going from (3.9-8) to (3.9-9). The result is

$$\Psi(\tau) = \bar{A}^2 + \int_0^\infty df_1 \int_0^\infty df_2 w(f_1)w(f_2)$$
$$\left[\frac{\cos 2\pi(f_1 + f_2)\tau}{\alpha^2 + [2\pi(f_1 + f_2)]^2} + \frac{\cos 2\pi(f_1 - f_2)\tau}{\alpha^2 + [2\pi(f_1 - f_2)]^2}\right]$$

It is convenient to define $w(-f)$ for negative frequencies to be equal to $w(f)$. The integration with respect to f_2 may then be taken from $-\infty$ to $+\infty$ and we get

$$\Psi(\tau) = \bar{A}^2 + \int_0^\infty df_1 \int_{-\infty}^{+\infty} df_2 w(f_1)w(f_2)\,\frac{\cos 2\pi(f_1 - f_2)\tau}{\alpha^2 + [2\pi(f_1 - f_2)]^2} \qquad (3.9\text{-}29)$$

The power spectrum $W(f)$ of $A(t)$ may be obtained by integrating $\Psi(\tau)$:

$$W(f) = 4\int_0^\infty \Psi(\tau)\cos 2\pi f\tau\,d\tau$$

Let us concern ourselves with the fluctuating portion $A(t) - \bar{A}$ of $A(t)$. Its power spectrum $W_c(f)$ is

$$W_c(f) = 4\int_0^\infty (\Psi(\tau) - \bar{A}^2)\cos 2\pi f\tau\,d\tau$$

The integration is simplified by using Fourier's integral formula in the form

$$\int_0^\infty d\tau \int_{-\infty}^{+\infty} df_2 F(f_2)\cos 2\pi(u - f_2)\tau = \tfrac{1}{2}F(u)$$

We get

$$\begin{aligned} W_c(f) &= \frac{1}{\alpha^2 + 4\pi^2 f^2}\int_0^\infty df_1[w(f_1)w(f + f_1) + w(f_1)w(-f + f_1)] \\ &= \frac{1}{\alpha^2 + 4\pi^2 f^2}\int_{-\infty}^{+\infty} w(f_1)w(f - f_1)\,df_1 \end{aligned} \qquad (3.9\text{-}30)$$

The simplicity of this result suggests that a simpler derivation may be found. If we attempt to use the result

$$w(f) = \underset{T\to\infty}{\text{Limit}}\ \frac{\overline{2|S(f)|^2}}{T} \qquad (2.5\text{-}3)$$

where $S(f)$ is given by (2.1–2) we find that we need the result

$$\underset{T\to\infty}{\text{Limit}}\ \frac{2}{T}\int_0^T dt_1 \int_0^T dt_2 e^{2\pi i f(t_2-t_1)} I^2(t_1)I^2(t_2) \\ = \int_{-\infty}^{+\infty} w(f_1)w(f-f_1)\,df_1 \tag{3.9–31}$$

where $f > 0$ and $I(t)$ is a noise current with $w(f)$ as its power spectrum. This may be proved by using (3.9–7) and

$$8\int_0^\infty \psi^2(\tau)\cos 2\pi f\tau\,d\tau = \int_{-\infty}^{+\infty} w(x)w(f-x)\,dx$$

which is given by equation (4C-6) in Appendix 4C.

An expression for the mean square fluctuation of $A(t)$ in terms of $w(f)$ may be obtained by setting τ equal to zero in (3.9–29)

$$\overline{(A(t)-\bar{A})^2} = \Psi(0) - \bar{A}^2 \\ = \int_0^\infty df_1 \int_{-\infty}^{+\infty} df_2 \frac{w(f_1)w(f_2)}{\alpha^2 + 4\pi^2(f_1-f_2)^2} \tag{3.9–32}$$

The same result may be obtained by integrating $W_c(f)$, (3.9–30), from 0 to ∞:

$$\int_0^\infty \frac{df}{\alpha^2+4\pi^2f^2}\int_{-\infty}^{+\infty} df_1 w(f_1)w(f-f_1) \tag{3.9–33}$$

Although this differs in appearance from (3.9–32) it may be transformed into that expression by making use of $w(-f) = w(f)$.

Suppose that $I(t)$ is the current through an ideal band pass filter so that $w(f)$ is zero except in the band $f_a < f < f_b$ where it is w_0. Then, if $3f_a > f_b$,

$$\bar{A} = \frac{w_0}{\alpha}(f_b - f_a) \tag{3.9–34}$$

$$\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx = \begin{cases} 2w_0^2(f_b - f_a - f) & 0 < f \le f_b - f_a \\ w_0^2(f - 2f_a) & 2f_a \le f \le f_b + f_a \\ w_0^2(2f_b - f) & f_b + f_a \le f \le 2f_b \end{cases}$$

and is zero outside these ranges. The power spectrum $W_c(f)$ may be obtained immediately from (3.9–30) by dividing these values by $\alpha^2 + 4\pi^2f^2$.

From (3.9–33)

$$\overline{(A(t)-\bar{A})^2} = 2w_0^2\int_0^{f_b-f_a}\frac{(f_b-f_a-f)\,df}{\alpha^2+4\pi^2f^2} \\ + w_0^2\int_{2f_a}^{f_b+f_a}\frac{(f-2f_a)}{\alpha^2+4\pi^2f^2}\,df + w_0^2\int_{f_b+f_a}^{2f_b}\frac{(2f_b-f)}{\alpha^2+4\pi^2f^2}\,df$$

If an exact answer is desired the integrations may be performed. When we assume that $f_b - f_a << f_b + f_a$ we may obtain approximations for the last two integrals.

$$\overline{(A(t) - \bar{A})^2} = w_0^2\left[\frac{f_b - f_a}{\pi\alpha}\tan^{-1}\frac{2\pi(f_b - f_a)}{\alpha} - \frac{1}{4\pi^2}\log\frac{\alpha^2 + 4\pi^2(f_b - f_a)^2}{\alpha^2} + \frac{(f_b - f_a)^2}{\alpha^2 + 4\pi^2(f_b + f_a)^2}\right]$$

Furthermore, if $2\pi(f_b - f_a)/\alpha$ is large we have

$$\overline{(A(t) - \bar{A})^2} = w_0^2\frac{f_b - f_a}{2\alpha}$$

and the relative r.m.s. fluctuation is

$$\text{r.m.s. of } \left[\frac{(A(t) - \bar{A})}{\bar{A}}\right] \approx \left[\frac{\alpha}{2(f_b - f_a)}\right]^{1/2}$$

This result may also be obtained from (3.9–10) and (3.9–11) by assuming α so small that the integral for $A(t)$ may be broken into a great many integrals each extending over an interval T. αT is assumed so small that $e^{-\alpha u}$ is substantially constant over each interval.

3.10 Distribution of Noise Plus Sine Wave

Suppose we have a steady sinusoidal current

$$I_p = I_p(t) = P\cos(\omega_p t - \varphi_p) \tag{3.10-1}$$

We pick times $t_1, t_2, \cdots$ at random and note the corresponding values of the current. How are these values distributed? Picking the times at random in (3.10-1) is the same, statistically, as holding t constant and picking the phase angles φ_p at random from the range 0 to 2π. If I_p be regarded as a random variable defined by the random variable φ_p, its characteristic function is

$$\begin{aligned}\text{ave. } e^{izI_p} &= \frac{1}{2\pi}\int_0^{2\pi} e^{izP\cos(\omega_p t - \varphi)}\,d\varphi \\ &= J_0(Pz)\end{aligned} \tag{3.10-2}$$

and its probability density is

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izI_p}J_0(Pz)\,dz = \begin{cases}\frac{1}{\pi}(P^2 - I_p^2)^{-1/2} & |I_p| < P \\ 0 & |I_p| > P\end{cases} \tag{3.10-3}$$

In this case it is simpler to obtain the probability density directly from (3.10-1) instead of from the characteristic function.

Now suppose that we have a noise current I_N plus a sine wave. By combining our representation (2.8-6) for I_N with the idea of φ_p being random mentioned above we are led to the representation

$$
\begin{aligned}
I(t) &= I = I_p + I_N \\
&= P \cos (\omega_p t - \varphi_p) + \sum_1^M c_n \cos (\omega_n t - \varphi_n), \qquad (3.10\text{-}4) \\
c_n^2 &= 2w(f_n)\Delta f
\end{aligned}
$$

where φ_p and $\varphi_1, \cdots \varphi_M$ are independent random angles.

If we note I at the random times $t_1, t_2 \cdots$ how are the observed values distributed? Since I_p and I_N may be regarded as independent random variables and since the characteristic function for the sum of two such variables is the product of their characteristic functions we have from (3.1-6) and (3.10-2)

$$
\begin{aligned}
\text{ave. } e^{izI} &= \text{ave. } e^{iz(I_p + I_N)} \\
&= J_0(Pz) \exp\left(\frac{-\psi_0 z^2}{2}\right) \qquad (3.10\text{-}5)
\end{aligned}
$$

which gives the characteristic function of I. The probability density of I is[37]

$$
\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izI-(\psi_0 z^2/2)} J_0(Pz)\, dz = \frac{1}{\pi\sqrt{2\pi\psi_0}} \int_0^\pi e^{-(I-P\cos\theta)^2/2\psi_0}\, d\theta \qquad (3.10\text{-}6)
$$

In the same way the two-dimensional probability density of (I_1, I_2), where $I_1 = I(t)$ is a sine wave plus noise (3.10-4) and $I_2 = I(t + \tau)$ is its value at a constant interval τ later, may be shown to be

$$
\frac{(\psi_0^2 - \psi_\tau^2)^{-1/2}}{2\pi} \int_0^{2\pi} d\theta \exp\left[-\frac{B(\theta)}{2(\psi_0^2 - \psi_\tau^2)}\right] \qquad (3.10\text{-}7)
$$

where

$$
\begin{aligned}
B(\theta) = \psi_0[(I_1 - P\cos\theta)^2 &+ (I_2 - P\cos(\theta + \omega_p\tau))^2] \\
&- 2\psi_\tau(I_1 - P\cos\theta)(I_2 - P\cos(\theta + \omega_p\tau))
\end{aligned}
$$

The characteristic function for I_1 and I_2 is

$$
\begin{aligned}
\text{ave. } e^{iuI_1 + ivI_2} &= J_0\left(P\sqrt{u^2 + v^2 + 2uv\cos\omega_p\tau}\right) \\
&\quad \times \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \qquad (3.10\text{-}8)
\end{aligned}
$$

[37] A different derivation of this expression is given by W. R. Bennett, *Jour. Acous. Soc. Amer.*, Vol. 15, p. 165 (Jan. 1944); *B.S.T.J.*, Vol. 23, p. 97 (Jan. 1944).

Sometimes the distribution of the envelope of

$$I = P \cos pt + I_N \tag{3.10-9}$$

is of interest. Here we have replaced ω_p by p and have set φ_p to zero. By the envelope we mean $R(t)$ given by

$$R^2(t) = R^2 = (P + I_c)^2 + I_s^2 \tag{3.10-10}$$

where I_c is the component of I_N "in phase" with cos pt and I_s is the component "in phase" with sin pt:

$$I_c = \sum c_n \cos [(\omega_n - p)t - \varphi_n]$$

$$I_s = \sum c_n \sin [(\omega_n - p)t - \varphi_n]$$

$$I_N = I_c \cos pt - I_s \sin pt$$

$$\overline{I_N^2} = \overline{I_c^2} = \overline{I_s^2} = \psi_0$$

Since I_c and I_s are distributed normally about zero with a variance of ψ_0, the probability densities of the variables

$$x = P + I_c$$

$$y = I_s$$

are

$$(2\pi\psi_0)^{-1/2} \exp - \frac{(x - P)^2}{2\psi_0}$$

$$(2\pi\psi_0)^{-1/2} \exp - \frac{y^2}{2\psi_0}$$

respectively. Setting

$$x = R \cos \theta$$

$$y = R \sin \theta$$

and using these distributions shows that the probability of a point (x, y) lying in the ring R, $R + dR$ is

$$\frac{R\,dR}{2\pi\psi_0} \int_0^{2\pi} \exp\left[-\frac{1}{2\psi_0}(R^2 + P^2 - 2RP\cos\theta)\right] d\theta$$

$$= \frac{R\,dR}{\psi_0} \exp\left[-\frac{R^2 + P^2}{2\psi_0}\right] I_0\left(\frac{RP}{\psi_0}\right) \tag{3.10-11}$$

where I_0 is the Bessel function with imaginary argument.

$$I_0(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n!\, n!}$$

and is a tabulated function. Thus (3.10-11) gives the probability density of the envelope R.

The average value of R^n may be obtained by multiplying (3.10-11) by R^n and integrating from 0 to ∞. Expansion of the Bessel function and termwise integration gives

$$\overline{R^n} = (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right)e^{-P^2/2\psi_0}\,{}_1F_1\left(\frac{n}{2}+1;\,1;\,\frac{P^2}{2\psi_0}\right)$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right){}_1F_1\left(-\frac{n}{2};\,1;\,-\frac{P^2}{2\psi_0}\right) \quad (3.10\text{-}12)$$

where ${}_1F_1$ is a hypergeometric function.[38] In going from the first line to the second we have used Kummer's first transformation of this function. A special case is

$$\overline{R^2} = P^2 + 2\psi_0 \quad (3.10\text{-}13)$$

When only noise is present, $P = 0$ and

$$\bar{R} = (2\psi_0)^{1/2}\Gamma(\tfrac{3}{2}) = \left(\frac{\psi_0\pi}{2}\right)^{1/2}$$
$$\overline{R^2} = 2\psi_0 \quad (3.10\text{-}14)$$

Before going further with (3.10-11) it is convenient to make the following change of notation

$$v = \frac{R}{\psi_0^{1/2}}, \qquad dv = \frac{dR}{\psi_0^{1/2}}, \qquad a = \frac{P}{\psi_0^{1/2}} \quad (3.10\text{-}15)$$

"a" is the ratio (sine wave amplitude)/(r.m.s. noise current).
Instead of the random variable R we now have the random variable v whose probability density is

$$p(v) = v\exp\left[-\frac{v^2+a^2}{2}\right]I_0(av) \quad (3.10\text{-}16)$$

Curves of $p(v)$ versus v are plotted in Fig. 6 for the values 0, 1, 2, 3, 5 of a. Curves showing the probability that v is less than a stated amount, i.e., distribution curves for v, are given in Fig. 7. These curves were obtained by integrating $p(v)$ numerically. The following useful expression for this probability has been given by W. R. Bennett in some unpublished work.

$$\int_0^v p(u)\,du = \exp\left[-\frac{v^2+a^2}{2}\right]\sum_{n=1}^{\infty}\left(\frac{v}{a}\right)^n I_n(av) \quad (3.10\text{-}17)$$

[38] Curves of this function are given in "Tables of Functions", Jahnke and Emde (1938), p. 275, and some of its properties are stated in Appendix 4C.

This is obtained by integration by parts using

$$\int u^n I_{n-1}(au)\, du = u^n I_n(au)/a$$

When $av >> 1$ but $1 << a - v$, Bennett has shown that (3.10–17) leads to

$$\int_0^v p(u)\, du \approx \left(\frac{v}{2\pi a}\right)^{1/2} \frac{1}{a - v} \exp\left[-\frac{(v-a)^2}{2}\right] \left(1 - \frac{3(a+v)^2 - 4v^2}{8av(a-v)^2} \cdots\right) \tag{3.10–18}$$

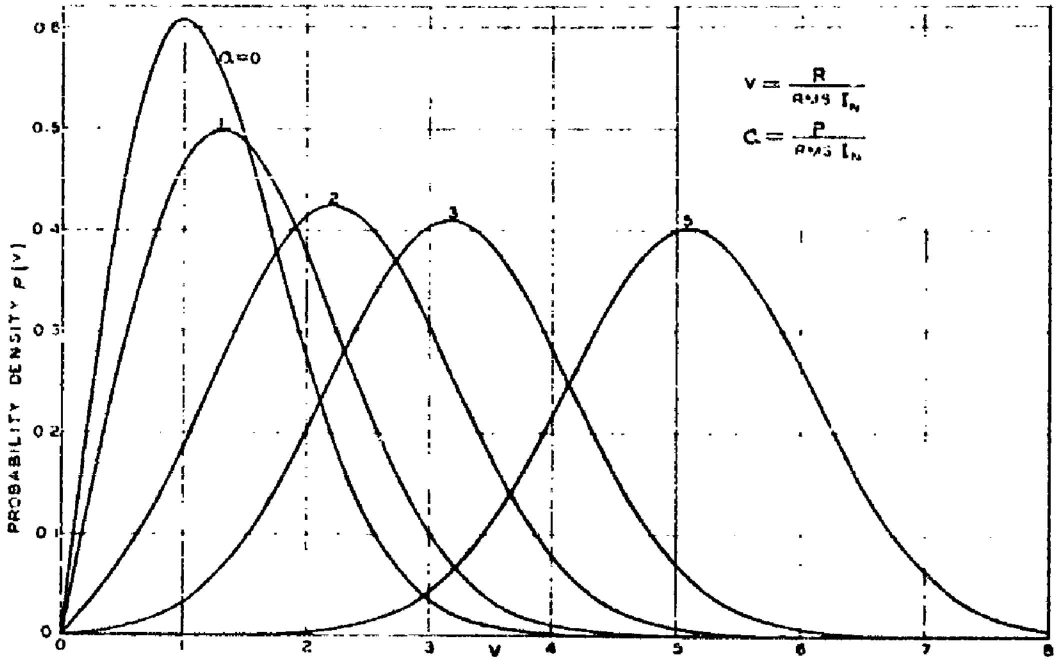

Fig. 6—Probability density of envelope R of $I(t) = P \cos pt + I_N$

This formula may also be obtained by putting the asymptotic expansion (3.10–19) for $p(v)$ in (3.10–17), integrating by parts twice, and neglecting higher order terms.

When av becomes large we may replace $I_0(av)$ by its asymptotic expression. The expression for $p(v)$ is then

$$p(v) \sim \left(1 + \frac{1}{8av}\right)\left(\frac{v}{2\pi a}\right)^{1/2} \exp\left[-\frac{(v-a)^2}{2}\right] \tag{3.10–19}$$

Thus when either a becomes large or v is far out on the tail of the probability density curve, the distribution behaves like a normal law. In terms of the original quantities, the normal law has an average of P and a standard deviation of $\psi_0^{1/2}$. This standard deviation is the same as the standard deviation of the instantaneous values of I_N. When $av \gg 1$ and $a \gg |v - a|$ we may expand the coefficient of the exponential term in (3.10-19) in powers of

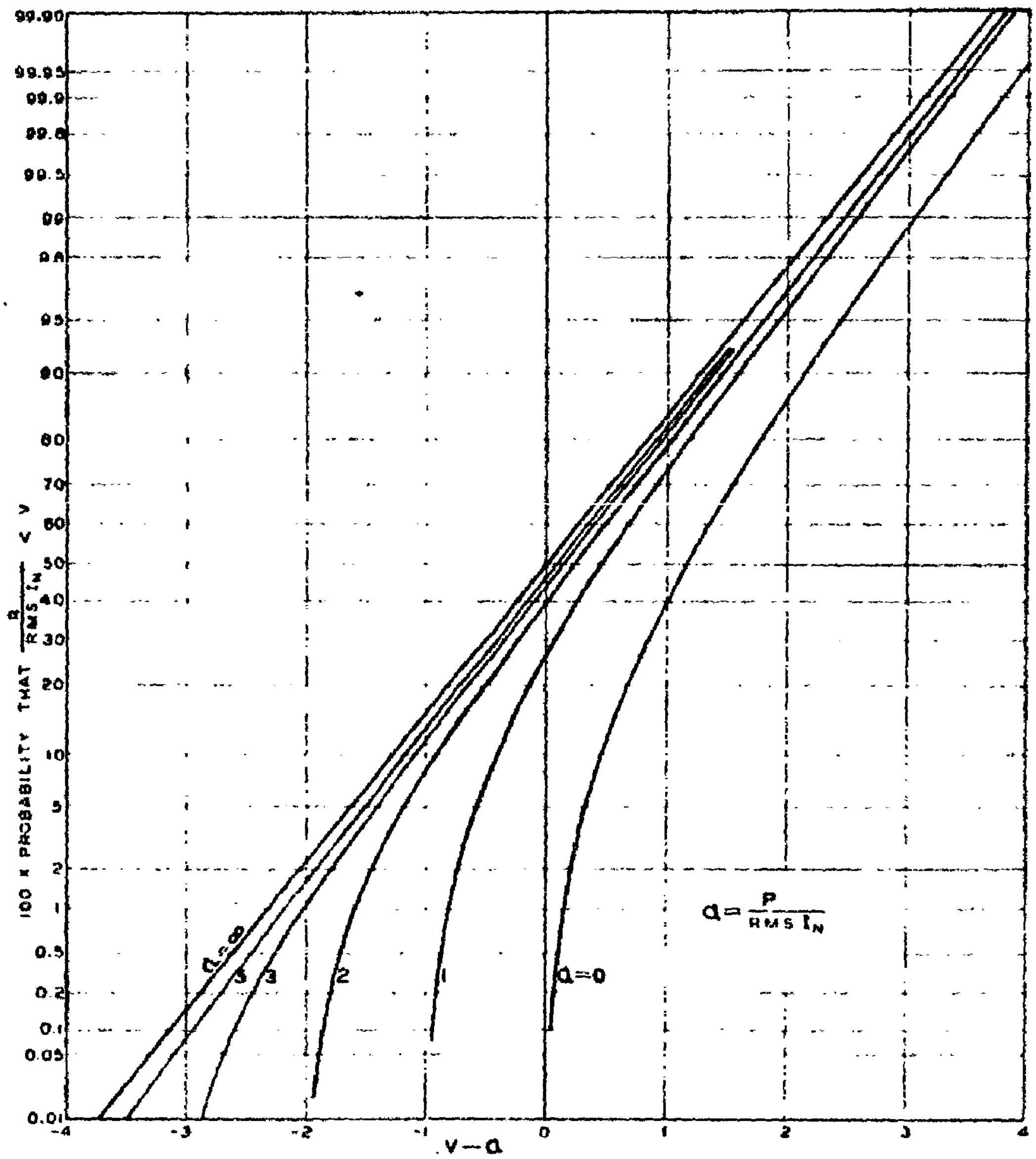

Fig. 7—Distribution function of envelope R of $I(t) = P \cos pt + I_N$

$(v - a)/a$. Integrating this expansion termwise gives, when terms of magnitude less than a^{-3} are neglected,

$$\int_0^v p(u)\,du \approx \frac{1}{2} + \frac{1}{2}\,erf\,\frac{v - a}{\sqrt{2}}$$

$$- \frac{1}{2a\sqrt{2\pi}}\left[1 - \frac{v - a}{4a} + \frac{1 + (v - a)^2}{8a^2}\right]\exp\left[-\frac{(v - a)^2}{2}\right]$$

When I consists of two sine waves plus noise

$$I = P \cos pt + Q \sin qt + I_N, \tag{3.10–20}$$

where the radian frequencies p and q are incommensurable, the probability density of the envelope R is

$$R \int_0^\infty r J_0(Rr) J_0(Pr) J_0(Qr) e^{-\psi_0 r^2/2} \, dr \tag{3.10–21}$$

where ψ_0 is $\overline{I_N^2}$. When Q is zero the integral may be evaluated to give (3.10–11). When both P and Q are zero the probability density for R when only noise is present is obtained. If there are three sine waves instead of two then another Bessel function must be placed in the integrand, and so on. To define R it is convenient to think of the noise as being confined to a relatively narrow band and the frequencies of the sine waves lying within, or close to, this band. As in equations (3.7–2) to (3.7–4), we refer all terms to a representative mid-band frequency $f_m = \omega_m/2\pi$ by using equations of the type

$$\begin{aligned} \cos pt &= \cos [(p - \omega_m)t + \omega_m t] \\ &= \cos (p - \omega_m)t \cos \omega_m t - \sin (p - \omega_m)t \sin \omega_m t. \end{aligned}$$

In this way we obtain

$$V = A \cos \omega_m t - B \sin \omega_m t = R \cos (\omega_m t + \theta) \tag{3.10–22}$$

where A and B are relatively slowly varying functions of t given by

$$\begin{aligned} A &= P \cos (p - \omega_m)t + Q \cos (q - \omega_m)t \\ &\qquad + \sum_n c_n \cos (\omega_n t - \omega_m t - \varphi_n) \\ B &= P \sin (p - \omega_m)t + Q \sin (q - \omega_m)t \\ &\qquad + \sum_n c_n \sin (\omega_n t - \omega_m t - \varphi_n) \end{aligned} \tag{3.10–23}$$

and

$$\begin{aligned} R^2 &= A^2 + B^2, \qquad R > 0 \\ \tan \theta &= B/A \end{aligned} \tag{3.10–24}$$

As might be expected, (3.10–21) is closely associated with the problem of random flights and may be obtained from Kluyver's result[39] by assuming

[39] G. N. Watson, "Theory of Bessel Functions" (Cambridge, 1922), p. 420.

the noise to correspond to a very large number of very small random displacements.

Another way of deriving (3.10–21) is to assume $(p - \omega_m)t$, $(q - \omega_m)t$, $\varphi_1, \varphi_2, \cdots$ are independent random angles. The characteristic function of A, B is

$$\text{ave. } e^{iuA+ivB} = J_0(P\sqrt{u^2+v^2})J_0(Q\sqrt{u^2+v^2})e^{-(\psi_0/2)(u^2+v^2)}$$

The probability density of A, B is

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, e^{-iuA-ivB} \text{ ave. } e^{iuA+ivB}$$

When the change of variables

$$A = R\cos\theta \qquad u = r\cos\varphi$$

$$B = R\sin\theta \qquad v = r\sin\varphi$$

is made the integration with respect to φ may be performed. The double integral becomes

$$\frac{1}{2\pi}\int_0^\infty rJ_0(Pr)J_0(Qr)J_0(Rr)e^{-(\psi_0/2)r^2}\, dr$$

This leads directly to (3.10–21) when we observe that $dAdB = RdRd\theta$.

Incidentally, if

$$I = Q(1 + k\cos pt)\cos qt + I_N$$

in which $p << q$, similar considerations show that the probability density of R is

$$\frac{R}{2\pi}\int_0^{2\pi} d\alpha \int_0^\infty rJ_0(Rr)J_0[Qr(1 + k\cos\alpha)]e^{-(\psi_0/2)r^2}\, dr$$

when ω_m is taken to be q. The integration with respect to r may be performed. This relation is closely connected with (3.10–11).

Returning now to the case in which I is the sum of two sine waves plus noise, we may show from (3.10–21) and

$$\int_0^\infty R^{n+1}J_0(Rr)\, dR = \frac{2^{n+1}\Gamma\left(1+\frac{n}{2}\right)}{r^{n+2}\Gamma\left(-\frac{n}{2}\right)}$$

that the average value of R^n is, when $-2 < re(n) < -\frac{1}{2}$,

$$\overline{R^n} = \frac{2^{n+1}\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \int_0^\infty r^{-n-1} J_0(Pr)J_0(Qr)e^{-\psi_0 r^2/2}\, dr$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_{k+m}(-x)^k(-y)^m}{k!k!m!m!} \quad (3.10\text{–}25)$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right)\sum_{k=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_k (y-x)^k}{k!k!} P_k\left(\frac{x+y}{x-y}\right)$$

It appears very probable that this result could be extended, by analytic continuation, to positive integer values of n. We have used the notation

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$$

$$x = \frac{P^2}{2\psi_0}, \qquad y = \frac{Q^2}{2\psi_0} \quad (3.10\text{–}26)$$

and have denoted the Legendre polynomial by $P_k(z)$. The series converge for all values of P, Q, and ψ_0 and terminate when n is an even positive integer.

When x or y, or both, are large in comparison with unity we may use the integral for $\overline{R^n}$ to obtain the asymptotic expansion, assuming $Q < P$ so that $y < x$,

$$\overline{R^n} \sim P^n \sum_{k=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_k\left(-\frac{n}{2}\right)_k}{k!x^k}\, {}_2F_1\left(k-\frac{n}{2}, k-\frac{n}{2}; 1; \frac{y}{x}\right) \quad (3.10\text{–}27)$$

When n is an even positive integer this series terminates and gives the same expression as (3.10–25). When n is an odd integer the ${}_2F_1$ may be expressed in terms of the complete elliptic functions E and K of modulus $y^{1/2}x^{-1/2}$:

$${}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right) = \frac{4}{\pi}E - \frac{2}{\pi}\left(1-\frac{y}{x}\right)K$$

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{y}{x}\right) = \frac{2}{\pi}K \quad (3.10\text{–}28)$$

The higher terms may be computed from

$$a(1-z)^2\,{}_2F_1(a+1, a+1; 1; z) = (2a-1)(1+z)\,{}_2F_1(a, a; 1; z)$$
$$+ (1-a)\,{}_2F_1(a-1, a-1; 1; z) \quad (3.10\text{–}29)$$

which is a special case of

$$ab(\gamma + 1)(1 - z)^2 {}_2F_1(a + 1, b + 1; c; z) = A\, {}_2F_1(a, b; c; z)$$
$$- (\gamma - 1)(c - a)(c - b)\, {}_2F_1(a - 1, b - 1; c; z) \quad (3.10\text{–}30)$$

where $\gamma = c - a - b$ and

$$A = (\gamma^2 - 1)\gamma + (1 - z)[(\gamma - 1)(c - b)(b - 1) + (\gamma + 1)a(c - a - 1)]$$

Although this expression does not show it, A is really symmetrical in a and b. A symmetrical form may be obtained by using the expression obtained by putting $z = 0$ in (3.10-30).

3.11 Shot Effect Representation

In most of the work in this part the representations (2.8-1) or (2.8-6) have been used as a starting point. Here we point out that the shot effect representation used in Part I may also be used as a starting point.

For example, suppose we wish to find the two dimensional distribution of $I(t)$ and $I(t + \tau)$ discussed in Section 3.2. This is a special case of the distribution of the two variables

$$I(t) = \sum_{k=-\infty}^{+\infty} F(t - t_k)$$
$$J(t) = \sum_{k=-\infty}^{+\infty} G(t - t_k) \quad (3.11\text{–}1)$$

where we now assume

$$\int_{-\infty}^{+\infty} F(t)\, dt = \int_{-\infty}^{+\infty} G(t)\, dt = 0 \quad (3.11\text{–}2).$$

in order that the average values of I and J may be zero. In fact, to get $I(t + \tau)$ from $J(t)$ we set $G(t)$ equal to $F(t + \tau)$.

The distribution of I and J may be obtained in much the same manner as was the distribution of I alone in section 1.4. The characteristic function of the distribution is

$$f(u, v) = \text{ave. } e^{iuI+ivJ}$$
$$= \exp \nu \int_{-\infty}^{+\infty} [e^{iuF(t)+ivG(t)} - 1]\, dt \quad (3.11\text{–}3)$$

where ν is the expected number of events (electron arrivals in the shot effect) per second. The probability density of I and J is

$$\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, e^{-iuI-ivJ} f(u, v) \quad (3.11\text{–}4)$$

The semi-invariants $\lambda_{m,n}$ are given by the generating function

$$\log f(u, v) = \sum_{m,n=1}^{k} \frac{\lambda_{m,n}}{m!\,n!} (iu)^m (iv)^n + o[(iu)^k, (iv)^k]$$

and are

$$\lambda_{m,n} = \nu \int_{-\infty}^{+\infty} F^m(t) G^n(t) \, dt \qquad (3.11\text{–}5)$$

As $\nu \to \infty$ the distribution of I and J approaches a two dimensional normal law. The approximation to this normal law may be obtained in much the same manner as in section 1.6. From our assumption (3.11–2) it follows that λ_{10} and λ_{01} are zero. From the relation between the second moments and semi-invariants λ we have

$$\begin{aligned} \mu_{11} &= \lambda_{20} + \lambda_{10}^2 = \nu \int_{-\infty}^{+\infty} F^2(t) \, dt \\ \mu_{12} &= \lambda_{11} + \lambda_{10}\lambda_{01} = \nu \int_{-\infty}^{+\infty} F(t)G(t) \, dt \\ \mu_{22} &= \lambda_{02} + \lambda_{01}^2 = \nu \int_{-\infty}^{+\infty} G^2(t) \, dt \end{aligned} \qquad (3.11\text{–}6)$$

where the notation in the subscripts of the μ's differs from that of the λ's, the change being made to bring it in line with sections 2.9 and 2.10 so that we may write down the normal distribution at once.

The formulas (3.11–6) are closely related to Rowland's generalization of Campbell's theorem mentioned just below equation (1.5–9).

4.0 Introduction

We shall consider two problems which concern noise passing through detectors or other non-linear devices. The first deals with the statistical properties of the output of a non-linear device, that is, with its average value, its fluctuation about this average and so on. The second problem may be stated more definitely: Given a non-linear device and an input consisting of noise alone, or of noise plus a signal. What is the power spectrum of the output?

There does not seem to be much published material on the first problem. However, from conversation with other people, I have learned that it has been studied independently by several investigators. The same is probably true of the second problem although here the published material is somewhat more plentiful. This makes it difficult to assign credit where credit is due. Much of the material given here had its origin in discussions with friends, especially with W. R. Bennett, J. H. Van Vleck, and David Middleton. Help was obtained from the recent paper[37] by Bennett, and also from the manuscript of a forthcoming paper by Middleton.[40]

4.1 Low Frequency Output of a Square Law Device

Let the output current I of the device be related to the input voltage V by

$$I = \alpha V^2 \tag{4.1-1}$$

where α is a constant. When the power spectrum of V is confined to a relatively narrow band, the power spectrum of I consists of two portions. One portion clusters around twice the mid-band frequency of V and the other around zero frequency. We are interested in the low frequency portion. The current corresponding to this portion will be denoted by I_{ll}, and is the current which would flow if a low pass filter were inserted in the output to remove the upper portion of the spectrum. It is convenient to divide I_{ll} into two components:

$$I_{ll} = I_{dc} + I_{lf} \tag{4.1-2}$$

[37] Loc. cit. (Section 3.10).

[40] Cruft Laboratory and the Research Laboratory of Physics, Harvard University, Cambridge, Mass. In the following sections references to Bennett's paper and Middleton's manuscript are made by simply giving the authors' names.

where the subscripts stand for "total low" frequency, "direct current." and "low frequency," respectively. We have

$$I_{dc} = \text{average } I_{tl} = \bar{I}_{tl} \tag{4.1-3}$$

$$\text{Mean Square } I_{lf} = \text{average } (I_{tl} - I_{dc})^2 = \overline{I_{tl}^2} - I_{dc}^2$$

Probably the simplest method of obtaining I_{dc} is to square the given expression for V and pick out the terms independent of time. Thus if

$$V = P \cos pt + Q \cos qt + V_N \tag{4.1-4}$$

we have

$$I_{dc} = \alpha\left(\frac{P^2}{2} + \frac{Q^2}{2} + \overline{V_N^2}\right) \tag{4.1-5}$$

I_{lf} may also be obtained by picking out the low frequency terms. However, here we wish to use the square law device, and the linear rectifier in the next section, to illustrate a general method of dealing with the statistical properties of the output of a non-linear device when the input voltage is restricted to a relatively narrow band.

If none of the low frequency spectrum is removed by filters,

$$I_{tl} = \alpha \frac{R^2}{2} \tag{4.1-6}$$

where R is the envelope of V. The probability density and the statistical properties of I_{tl} may be derived from this relation when the distribution function of R is known.[41] Before discussing these properties we shall establish (4.1-6).

Equation (4.1-6) is a special case of a more general result established in Section 4.3. However, its truth may be seen by taking the example

$$V = P \cos pt + Q \cos qt + V_N \tag{4.1-4}$$

where $f_p = p/2\pi$ and $f_q = q/2\pi$ lie within, or close to, the band of the noise voltage V_N.

By using formulas of the type

$$\begin{aligned} \cos pt &= \cos[(p - \omega_m)t + \omega_m t] \\ &= \cos(p - \omega_m)t \cos \omega_m t - \sin(p - \omega_m)t \sin \omega_m t \end{aligned} \tag{4.1-7}$$

[41] When part of the low-frequency spectrum is removed, the problem becomes much more difficult. I_{dc} may be obtained as above, but to get $\overline{I_{lf}^2}$ it is necessary to first determine the power spectrum of I (Section 4.5) and then integrate over the appropriate portion of it. Concerning the distribution of I_{lf}, our present knowledge tells us only that it lies between the one given by (4.1-6) and the normal law which it approaches when only a narrow portion of the low frequency spectrum is passed by the audio frequency filter (Section 4.3).

we may refer all terms to the mid-band frequency $f_m = \omega_m/2\pi$, as is done in equations (3.7–2) to (3.7–4).

In this way we obtain

$$V = A \cos \omega_m t - B \sin \omega_m t = R \cos (\omega_m t + \theta), \tag{4.1–8}$$

where A and B are relatively slowly varying functions of t given by

$$A = P \cos (p - \omega_m)t + Q \cos (q - \omega_m)t + \sum_n c_n \cos (\omega_n t - \omega_m t - \varphi_n),$$

$$B = P \sin (p - \omega_m)t + Q \sin (q - \omega_m)t + \sum_n c_n \sin (\omega_n t - \omega_m t - \varphi_n)$$

and

$$R^2 = A^2 + B^2, \qquad R > 0$$
$$\tan \theta = B/A. \tag{4.1–9}$$

This definition of R has also been given in equations (3.10–22, 23, 24).

The envelope of V is R and the output current is

$$I = \alpha R^2 \left[\frac{1}{2} + \frac{1}{2} \cos (2\omega_m t + 2\theta)\right] \tag{4.1–10}$$

Since R is a slowly varying function of time, so is R^2. The power spectrum of R^2 is confined to frequencies much lower than $2f_m$ and consequently the power spectrum of $R^2 \cos (2\omega_m t + 2\theta)$ is clustered around $2f_m$. Thus the only term in I contributing to the low frequency output is $\alpha R^2/2$ which is what we wished to show.

We now return to the statistical properties of I_{lf}. First, consider the case in which V consists of noise only, $V = V_N$, so that the probability density of the envelope R is

$$\frac{R}{\psi_0} e^{-R^2/2\psi_0} \tag{3.7–10}$$

where

$$\psi_0 = [rms\ V_N]^2 = \overline{V_N^2} \tag{4.1–11}$$

Hence

$$I_{dc} = \overline{I}_{lf} = \frac{\alpha \overline{R^2}}{2}$$
$$= \int_0^\infty \frac{\alpha R^2}{2} \frac{R}{\psi_0} e^{-R^2/2\psi_0} \, dR$$
$$= \alpha\psi_0$$

$$\overline{I_{lf}^2} = \overline{I_{lf}^2} - I_{dc}^2 = \int_0^\infty \frac{\alpha^2 R^5}{4\psi_0} e^{-R^2/2\psi_0} \, dR - I_{dc}^2$$
$$= \alpha^2\psi_0^2 \tag{4.1–12}$$

Second, consider the case in which

$$V = V_N + P \cos pt \tag{4.1-13}$$

where $p/2\pi$ lies near the noise band of V_N. The probability density of the envelope R is

$$\frac{R}{\psi_0} \exp\left[-\frac{R^2 + P^2}{2\psi_0}\right] I_0\left(\frac{RP}{\psi_0}\right) \tag{3.10-11}$$

From this and equations (3.10-12), (3.10-13), we find

$$I_{dc} = \frac{\alpha \overline{R^2}}{2} = \alpha\psi_0 + \frac{\alpha P^2}{2} \tag{4.1-14}$$

$$\overline{I_{\ell\ell}^2} = \frac{\alpha^2}{4} \overline{R^4} = \alpha^2 \left[2\psi_0^2 + 2P^2\psi_0 + \frac{P^4}{4}\right]$$

$$\overline{I_{\ell f}^2} = \overline{I_{\ell\ell}^2} - I_{dc}^2 = \alpha^2[\psi_0 + P^2]\psi_0 \tag{4.1-15}$$

In (4.1-14) ψ_0 is the mean square value of V_N and $P^2/2$ is the mean square value of the signal. These two equations show that I_{dc} and the rms value of $I_{\ell f}$ are independent of the distribution of the noise power spectrum in V_N as long as the input V is confined to a relatively narrow band. In other words, although this distribution does affect the power spectrum of the output, it does not affect the d.c. and rms $I_{\ell f}$ when ψ_0 and P are given. That the same is also true for a large class of non-linear devices was first pointed out by Middleton (see end of Section 4.9).

When the voltage is[42]

$$V = V_N + P \cos pt + Q \cos qt, \tag{4.1-4}$$

$p \neq q$, we obtain from equation (3.10-25)

$$I_{dc} = \frac{\alpha}{2} \overline{R^2} = \alpha\left(\psi_0 + \frac{P^2}{2} + \frac{Q^2}{2}\right)$$

$$\overline{I_{\ell\ell}^2} = \frac{\alpha^2}{4} \overline{R^4} \tag{4.1-16}$$

$$\overline{I_{\ell f}^2} = \alpha^2\left[\psi_0^2 + P^2\psi_0 + Q^2\psi_0 + \frac{P^2Q^2}{2}\right]$$

[42] These results are special cases, obtained by assuming no audio frequency filter, of formulas given by F. C. Williams, *Jour. Inst.* of *E. E.*, 80 (1937), 218–226. Williams also discusses the response of a linear rectifier to (4.1-4) when $P \gg Q + V_N$. An account of Williams' work is given by E. B. Moullin, "Spontaneous Fluctuations of Voltage," Oxford (1938), Chap. 7.

4.2 Low Frequency Output of a Linear Rectifier

In the case of the linear rectifier

$$I = \begin{cases} 0, & V < 0 \\ \alpha V, & V > 0 \end{cases} \tag{4.2-1}$$

the low frequency output current, assuming no audio frequency filter, is

$$I_{lf} = \frac{\alpha R}{\pi} \tag{4.2-2}$$

This formula, like its analogue (4.1-6) for the square law device, assumes that the applied signal and noise lie within a relatively narrow band. It may be used to compute the probability density and statistical properties of I_{lf} when the corresponding information regarding the envelope R of the applied voltage is known.

The truth of (4.2-2) may be seen by considering the output I. It consists of the positive halves of the oscillations of αV. The envelope of I is the same as that of αV. However, the area under the loops of I is only about $1/\pi$ of the area under αR, this being the ratio of the area under a loop of $\sin x$ to the area of a rectangle of unit height and length 2π. From the low frequency point of view these loops of I merge into a current which varies as $\alpha R/\pi$.

When V is a sine wave plus noise,

$$V = V_N + P \cos pt \tag{4.1-13}$$

the average value of I_{lf} is[43]

$$\begin{aligned} I_{dc} &= \frac{\alpha}{\pi} \bar{R} = \alpha \left(\frac{\psi_0}{2\pi}\right)^{1/2} {}_1F_1\left(-\frac{1}{2}; 1; -\frac{P^2}{2\psi_0}\right) \\ &= \alpha \left(\frac{\psi_0}{2\pi}\right)^{1/2} e^{-x/2}\left[(1 + x)I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right] \end{aligned} \tag{4.2-3}$$

where I_0, I_1 are Bessel functions of imaginary argument and

$$x = \frac{P^2}{2\psi_0} = \frac{\text{ave. sine wave power}}{\text{ave. noise power}} \tag{4.2-4}$$

[43] This result was discovered independently by several investigators, among whom we may mention W. R. Bennett and D. O. North. The latter has applied it to noise measurement work. He has found that the diode detector, when adapted to noise metering, is a great improvement over the thermocouple, and has used noise meters of this type satisfactorily since 1940. See D. O. North, "The Modification of Noise by Certain Non-Linear Devices", Paper read before I.R.E., Jan. 28, 1944.

ψ_0 being the average value of V_N^2. Equation (4.2-3) follows from the formulas (3.10-12) and (4B-9). When x is large the asymptotic expansion (4B-3) of the $_1F_1$ gives

$$I_{dc} \sim \frac{\alpha}{\pi}\left[P + \frac{\psi_0}{2P} + \frac{\psi_0^2}{8P^3} + \cdots\right] \tag{4.2-5}$$

Similarly, the mean square value of I_{tt} is

$$\overline{I_{tt}^2} = \frac{\alpha^2}{\pi^2}\overline{R^2} = \frac{\alpha^2}{\pi^2}(P^2 + 2\psi_0) \tag{4.2-6}$$

and the mean square value of the low frequency current I_{tf}, excluding the d.c., is given by

$$\overline{I_{tf}^2} = \overline{I_{tt}^2} - I_{dc}^2$$

When x is large we have

$$\overline{I_{tf}^2} \sim \frac{\alpha^2}{\pi^2}\left[\psi_0 - \frac{\psi_0^2}{2P^2}\cdots\right] = \frac{\alpha^2}{\pi^2}\psi_0\left[1 - \frac{1}{4x}\cdots\right] \tag{4.2-7}$$

and when $x = 0$,

$$\overline{I_{tf}^2} = \frac{\alpha^2}{\pi^2}\psi_0\left(2 - \frac{\pi}{2}\right) \tag{4.2-8}$$

Curves for I_{dc} are given in Figures 1, 2 and 3 of Bennett's paper. He also gives curves, in Fig. 4, showing $\overline{I_{tf}^2}$ versus x. These show that the effect of the higher order modulation terms is small when I_{tf} is computed by adding low frequency modulation products.

When V consists of two sine waves plus noise,

$$V = V_N + P\cos pt + Q\cos qt, \tag{4.1-4}$$

the average value of I_{tt} is, from (3.10-25), a sort of double $_1F_1$ function:

$$\begin{aligned} I_{dc} &= \frac{\alpha}{\pi}\bar{R} = \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\frac{1}{2})_{k+m}}{k!k!m!m!}(-x)^k(-y)^m \\ &= \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2})_k}{k!k!}(y-x)^k P_k\left(\frac{x+y}{x-y}\right) \end{aligned} \tag{4.2-9}$$

where

$$x = \frac{P^2}{2\psi_0}, \qquad y = \frac{Q^2}{2\psi_0}, \qquad P_k(z) = \text{Legendre polynomial} \tag{4.2-10}$$

If x is large and $y < x$, we have from (3.10-27) the asymptotic expression

$$I_{dc} \sim \frac{\alpha}{\pi}P\sum_{k=0}^{\infty}\frac{(-\frac{1}{2})_k(-\frac{1}{2})_k}{k!x^k}\,{}_2F_1\left(k-\tfrac{1}{2}, k-\tfrac{1}{2}; 1; \frac{y}{x}\right) \tag{4.2-11}$$

The ${}_2F_1$ may be expressed in terms of the complete elliptic functions E and K of modulus $y^{1/2}x^{-1/2}$. Thus

$$\begin{aligned} {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right) &= \frac{4}{\pi} E - \frac{2}{\pi}\left(1 - \frac{y}{x}\right) K, \\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{y}{x}\right) &= \frac{2}{\pi} K \end{aligned} \tag{3.10–28}$$

and the higher terms may be computed from the recurrence relation (3.10–29). The first term, $k = 0$, in (4.2–11) gives I_{dc} when the noise is absent.[44]

The mean square value of I_{lf} is

$$\overline{I_{lf}^2} = \frac{\alpha^2}{\pi^2} R^2 = \frac{\alpha^2}{\pi^2} [2\psi_0 + P^2 + Q^2] \tag{4.2–14}$$

From this expression and our expression for I_{dc}, the rms value of the low frequency current, I_{lf}, excluding the d.c., may be computed. For example, when the noise is small,

$$\begin{aligned} \overline{I_{lf}^2} \sim \frac{\alpha^2}{\pi^2}\Bigg[P^2 + Q^2 - \left(P\,{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right)\right)^2 \\ + 2\psi_0\left(1 - {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right)\frac{K}{\pi}\right)\Bigg] \end{aligned} \tag{4.2–15}$$

The term independent of ψ_0 gives the mean square low frequency current in the absence of noise. As Q goes to zero (4.2–15) approaches the leading term in (4.2–7), as it should. When $P = Q$ our formula breaks down and it appears that we need the asymptotic behavior of[45]

$$I_{dc} = \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(2k)!}{[k!]^4} (-x)^k$$

In view of the questionable nature of the derivation given in Section 3.10 of equations (4.2–9) and (4.2–11) it was thought that a numerical check on their equivalence would be worth while. Accordingly, the values $x = 4$, $y = 3$ were used in the second series of (4.2–9). It was found that the largest term (about 130) in the summation occurred at $k = 11$. In all, 24 terms were taken. The result obtained was

$$\frac{\bar{R}}{\sqrt{2\psi_0}} = 2.5502$$

[44] See W. R. Bennett, *B.S.T.J.*, Vol. 12 (1933), 228–243.
[45] This may be done by the method given by W. B. Ford, *Asymptotic Developments*, Univ. of Mich. Press (1936), Chap. VI.

For the same values of x and y the asymptotic series (4.2–11) gave

$$2.40 + 0.171 + .075 + 0.52 + \cdots$$

If we stop just before the smallest term we get 2.57 for the sum. If we include the smallest term we get 2.65. This agreement indicates that (4.2–11) is actually the asymptotic expansion of (4.2–9).

When the voltage is of the form

$$V = Q(1 + k\cos pt)\cos qt + V_N$$

we may use

$$\overline{R^n} = (2\psi_0)^{n/2}\Gamma\left(1 + \frac{n}{2}\right)\frac{1}{2\pi}\int_0^{2\pi} {}_1F_1\left[-\frac{n}{2}; 1; -y(1 + k\cos\theta)^2\right]d\theta \tag{4.2–16}$$

where R is the envelope with respect to the frequency $q/2\pi$ and y is given by (4.2–10). The integral may be evaluated by writing ${}_1F_1$ as a power series and integrating termwise using the result

$$\frac{1}{2\pi}\int_0^{2\pi}(1 + k\cos\theta)^{\ell}\cos m\theta\, d\theta = \frac{(-\ell)_m}{2^m m!}(-k)^m {}_2F_1\left[\frac{m-\ell}{2}, \frac{m-\ell+1}{2}; m+1; k^2\right] \tag{4.2–17}$$

where m is a non-negative integer, ℓ any number,

$$(\alpha)_m = \alpha(\alpha + 1)\cdots(\alpha + m - 1), \qquad (\alpha)_0 = 1, \qquad \text{and} \qquad (0)_0 = 1.$$

The integral may also be evaluated in terms of the associated Legendre function.

By applying the methods of Section 3.10 to (4.2–16) we are led to

$$\begin{aligned}\overline{R^2} &= Q^2\left(1 + \frac{k^2}{2}\right) + 2\psi_0 \\ \bar{R} &\sim Q\sum_{s=0}^{\infty}\frac{(-\frac{1}{2})_s(-\frac{1}{2})_s}{s!\,y^s}\,{}_2F_1(s - \tfrac{1}{2}, s; 1; k^2)\end{aligned} \tag{4.2–18}$$

where the asymptotic series holds when y is very large and k is not too close to unity. These expressions give

$$\overline{I_{lf}^2} \sim \frac{\alpha^2}{\pi^2}\left(Q^2\frac{k^2}{2} + \psi_0[2 - (1 - k^2)^{-1/2}] + \cdots\right) \tag{4.2–19}$$

The reader might be tempted to associate the coefficient of ψ_0 in (4.2-19) with the continuous portion of the output power spectrum. However, this would not be correct. It appears that the principal contribution of the continuous portion of the power spectrum to $\overline{I_{lf}^2}$ is $\alpha^2\psi_0/\pi^2$, just as in (4.2-7) when k is zero. The difference between this and the corresponding term in (4.2-19) seems to arise from the fact that the amplitude of the recovered signal is not exactly $\alpha Qk/\pi$ but is modified by the presence of the noise. This general type of behavior might be expected on physical grounds since changing P, say doubling it, in (4.2-7) does not appreciably affect the $\overline{I_{lf}^2}$ in (4.2-7) (which is due entirely to the continuous portion of the noise spectrum). The modulating wave may be regarded as slowly making changes of this sort in P.

4.3 Some Statistical Properties of the Output of a General Non-Linear Device

Our general problem is this: Given a non-linear device whose output I is related to its input V by the relation

$$I = \frac{1}{2\pi}\int_C F(iu)e^{iVu}\,du \tag{4A-1}$$

which is discussed in Appendix 4A. Let the input V contain noise in addition to the signal. Choose some frequency band in the output for study. What are the statistical properties of the current flowing in this band?

It seems to be difficult to handle this general problem. However, it appears that the two following results are true.

1. As the output band is chosen narrower and narrower the statistical properties of the corresponding current approach those of the random noise current discussed in Part III (provided no signal harmonic lies within the band). In particular, the instantaneous current values are distributed normally.

2. When the input V is confined to a relatively narrow band the power spectrum of the output I is clustered around the 0^{th} (d.c.), 1st, 2nd, etc. harmonics of the midband frequency of V. The low frequency output including the d.c. is

$$I_{lf} = A_0(R) = \frac{1}{2\pi}\int_C F(iu)J_0(uR)\,du \tag{4.3-11}$$

where R is the envelope of V.

The envelope of the nth harmonic of the output, when $n > 0$, is

$$A_n(R) = \frac{1}{\pi}\int_C F(iu)J_n(uR)\,du \tag{4.3-1}$$

The mathematical statement is

$$I = \sum_{n=0}^{\infty} A_n(R) \cos (n\omega_m t + n\theta) \qquad (4.3\text{-}9)$$

where $f_m = \omega_m/(2\pi)$ is the representative mid-band frequency of V and θ is a relatively slowly varying phase angle. The results of Sections 4.1 and 4.2 are special cases of this.

Middleton's result that the noise power in each of the output bands (in the entire band corresponding to a given harmonic) depends only on $\overline{V_N^2} = \psi_0$ and not on the spectrum of V_N, where V_N is the noise voltage component of V, may also be obtained from (4.3-9). We note that the total power in the n^{th} band depends only on the mean square value of its envelope $A_n(R)$, and that the probability density of the envelope R of the input involves V_N only through ψ_0.

The argument we shall use in discussing the first result is not very satisfactory. It runs as follows. The output current I may be divided into two parts. One consists of sinusoidal terms due to the signal. The other consists of noise. We shall be concerned only with the latter which we shall call I_N. The correlation between two values of I_N separated by an interval of time approaches zero as the interval becomes large. Let τ be an interval long enough to ensure that the two values of I_N are substantially independent. Choose an interval of time T long enough to contain many intervals of length τ. Expand I_N as a Fourier series over this interval. We have

$$\begin{aligned} I_N &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right] \\ a_n - i b_n &= \frac{2}{T} \int_0^T e^{-i2\pi n t/T} I_N(t)\, dt \end{aligned} \qquad (4.3\text{-}2)$$

Let the band chosen for study be $f_0 - \frac{\beta}{2}$ to $f_0 + \frac{\beta}{2}$ and let

$$T\left(f_0 - \frac{\beta}{2}\right) = n_1, \qquad T\left(f_0 + \frac{\beta}{2}\right) = n_2 \qquad (4.3\text{-}3)$$

where n_1 and n_2 are integers. The number of components in the band is $(n_2 - n_1)$. We suppose β is such that this is small in comparison with T/τ. The output of the band is

$$J_N = \sum_{n=n_1}^{n_2} \left[a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n t}{T} \right] \qquad (4.3\text{-}4)$$

where

$$a_n - ib_n = \frac{2}{T}\int_0^T e^{-i2\pi((n/T)-f_0)t} e^{-i2\pi f_0 t} I_N(t)\, dt$$

$$n = \frac{n_1 + n_2}{2} + n - \frac{n_1 + n_2}{2} = f_0 T + (n - f_0 T) \tag{4.3-5}$$

We choose the band so narrow that

$$n_2 - n_1 \ll T/\tau \quad \text{or} \quad \beta\tau \ll 1 \tag{4.3-6}$$

This enables us to write approximately

$$a_n - ib_n = \sum_{r=1}^{r_1} e^{-i2\pi((n/T)-f_0)r\tau} \frac{2}{T}\int_{(r-1)\tau}^{r\tau} e^{-i2\pi f_0 t} I_N(t)\, dt$$

$r_1 = T/\tau$, T being chosen to make r_1 an integer. Suppose we do this for a large number of intervals of length T. Then $I_N(t)$ will differ from interval to interval. The set of integrals for $r = 1$ gives us an array of values which we regard as defining the distribution of a complex random variable, say x_1. Similarly the set of integrals for $r = 2$ defines the distribution of a second random variable x_2, and so on to x_{r_1}. Because we have chosen τ so large that $I_N(t)$ in any one integral is practically independent of its values in the other integrals we may say that $x_1, x_2, \cdots x_{r_1}$ are independent. We have

$$a_{n_1} - ib_{n_1} = \sum_{r=1}^{r_1} e^{-i2\pi((n/T)-f_0)r\tau} x_r$$

$$a_{n_1+1} - ib_{n_1+1} = \sum_{r=1}^{r_1} e^{-i2\pi((n_1+1/T)-f_0)r\tau} x_r$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n_2} - ib_{n_2} = \sum_{r=1}^{r_1} e^{-i2\pi((n_2/T)-f_0)r\tau} x_r$$

and if $n_2 - n_1 \ll r_1$, as was assumed in (4.3-6), we may apply the central limit theorem to show that $a_{n_1}, b_{n_1}, a_{n_1+1}, \cdots a_{n_2}, b_{n_2}$ tend to become independent and normally distributed about zero as we let the band width $\beta \to 0$ and $T \to \infty$ (and hence $r_1 \to \infty$) in such a way as to keep $n_2 - n_1$ fixed. In this work we make use of the fact that $I_N(t)$ is such that the real and imaginary parts of $x_1, x_2, \cdots x_r$ all have the same average and standard deviation. It is convenient to assume $f_0 T$ is an integer.

Thus as the band width β approaches zero the band output J_N given by (4.3-4) may be represented in the same way, namely as (2.8-1), as was the random noise current studied in Part III. Hence J_N tends to have the same properties as the random noise current studied there. For example, the distribution of J_N tends towards a normal law. In our discussion we had to assume that $\beta\tau \ll 1$. If the voltage V applied to the non-linear device is confined to a relatively narrow frequency band, say $f_b - f_a$, it appears that the interval τ (chosen above so that $I(t)$ and $I(t + \tau)$ are substantially independent) may be taken to be of the order of $1/(f_b - f_a)$. In this case J_N tends to behave like a random noise current if $\beta/(f_b - f_a)$ is much smaller than unity.

We now turn our attention to the second statement made at the beginning of this section. Let the applied voltage be confined to a relatively narrow band so that it may be represented by equation (4.1-8) of Section 4.1,

$$V = R \cos (\omega_m t + \theta), \qquad R \geq 0, \tag{4.1-8}$$

where $f_m = \omega_m/(2\pi)$ is some representative frequency within the band and R and θ are functions of time which vary slowly in comparison with $\cos \omega_m t$. We call R the envelope of V.

From equation (4A 1)

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iuR \cos (\omega_m t + \theta)} \, du \tag{4.3-7}$$

We expand the integrand by means of

$$e^{iz \cos \varphi} = \sum_{n=0}^{\infty} \epsilon_n i^n \cos n\varphi J_n(z) \tag{4.3-8}$$

where ϵ_0 is 1 and ϵ_n is 2 when $n > 0$ and $J_n(x)$ is a Bessel function. Thus

$$I = \sum_{n=0}^{\infty} A_n(R) \cos (n\omega_m t + n\theta) \tag{4.3-9}$$

where

$$A_n(R) = \epsilon_n \frac{i^n}{2\pi} \int_C F(iu) J_n(uR) \, du \tag{4.3-10}$$

Since R is a relatively slowly varying function of time we expect the same to be true of $A_n(R)$, at least for moderately small values of n. Thus from (4.3-9) we see that the power spectrum of I will consist of a succession of bands, the n^{th} band being clustered around the frequency nf_m. If we eliminate all of the bands except the n^{th} by means of a filter we see that the output will have the envelope $A_n(R)$ when $n \geqq 1$. Taking n to be zero, shows that the low frequency output is simply

$$A_0(R) = \frac{1}{2\pi} \int_C F(iu) J_0(uR) \, du \tag{4.3-11}$$

Taking n to be one shows that the band around f_m is given by

$$\frac{A_1(R)}{R} V \tag{4.3-12}$$

The statistical properties of the low frequency output and of the envelopes of the output bands may be obtained from those of R. For example, the probability density of $A_n(R)$ is of the form

$$p(R) \Big/ \frac{dA_n(R)}{dR} \tag{4.3-13}$$

where $p(R)$ is the probability density of R. In this expression R is considered as a function of A_n.

It should be noted that we have been assuming that all of the band surrounding the harmonic frequency nf_m is taken. When we take only a portion of it, presumably the statistical properties will tend to approach those of a random noise current in accordance with the first statement made at the beginning of this section.

When we apply (4.3-11) to the square law device we have

$$F(iu) = \frac{2\alpha}{(iu)^3}$$

$$A_0(R) = -\frac{2\alpha}{2\pi i} \int^{(0+)} \frac{J_0(uR)}{u^3}\, du$$

$$= \frac{\alpha}{2} R^2$$

When we apply (4.3-11) to the linear rectifier:

$$F(iu) = -\frac{\alpha}{u^2}$$

$$A_0(R) = -\frac{\alpha}{2\pi} \int_{-\infty}^{+\infty} \frac{J_0(uR)}{u^2}\, du = \frac{\alpha R}{\pi}$$

where the path of integration passes under the origin. These two results agree with those obtained in Section 4.1 and 4.2 from simple considerations.

As a final example we find the low frequency output of a biased linear rectifier in terms of the envelope R of the applied voltage. From the table of $F(iu)$ given in Appendix 4A we see that $F(iu)$ corresponding to

$$I = 0, \qquad V < B$$

$$I = V - B, \qquad V > B$$

is

$$F(iu) = -\frac{e^{-iuB}}{u^2}$$

Consequently, the low frequency output is

$$A_0(R) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iuB}J_0(uR)u^{-2}\,du$$

where the path of integration is indented downwards at the origin. When $B > R$ the value of the integral is zero since then the path of integration may be closed in the lower half plane by an infinite semi-circle This value also follows at once from the physics of the problem. When $-R < B < R$ we may integrate by parts and get

$$\begin{aligned} A_0(R) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iuB}[iBJ_0(uR) + RJ_1(uR)]u^{-1}\,du \\ &= -\frac{B}{2} + \frac{1}{\pi}\int_0^{\infty} [B\sin uBJ_0(uR) + R\cos uBJ_1(uR)]u^{-1}\,du \\ &= -\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2} \\ &= -\frac{B}{2} + \frac{R}{\pi}F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \frac{B^2}{R^2}\right), \quad -R < B < R \end{aligned} \tag{4.3-14}$$

This hypergeometric function turns up again in equation (4.7-6). Also in the range $-R < B < R$,

$$\frac{dA_0}{dR} = \frac{1}{\pi}\sqrt{1 - \frac{B^2}{R^2}}$$

When B is negative and $R < -B$, the path of integration may be closed by an infinite semicircle in the upper half plane and the value of the integral is proportional to the residue of the pole at the origin:

$$\begin{aligned} A_0(R) &= 2\pi i\left(-\frac{1}{2\pi}\right)(-iB) \\ &= -B \end{aligned}$$

Thus, to summarize, the low frequency output for our linear rectifier is, for $B > 0$, (R is always positive)

$$\begin{aligned} &A_0(R) = 0, \qquad R < B \\ &A_0(R) = -\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}, \qquad B < R \end{aligned} \tag{4.3-15}$$

and for $B < 0$ it is

$$A_0(R) = |B|, \quad R < |B|$$

$$A_0(R) = +\frac{|B|}{2} + \frac{|B|}{\pi}\arcsin\frac{|B|}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}, \quad |B| < R \tag{4.3-16}$$

where the arc sines lie between 0 and $\pi/2$. $A_0(R)$ and its first derivative with respect to R are continuous.

From (4.3-15), the d.c. output current is, for $B > 0$,

$$I_{dc} = \int_B^\infty \left[-\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}\right] p(R)\, dR \tag{4.3-15}$$

where $p(R)$ is the probability density of the envelope of the input V, e.g., $p(R)$ is of the form (3.7-10) for noise alone, and of the form (3.10-11) for noise plus a sine wave. Similarly, the rms value of the low frequency current I_{lf}, excluding d.c., may be computed from

$$\overline{I_{lf}^2} = \overline{I_{lt}^2} - I_{dc}^2$$

where, if $B > 0$,

$$\overline{I_{lt}^2} = \int_B^\infty \left[-\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}\right]^2 p(R)\, dR \tag{4.3-16}$$

If V consists of a sine wave of amplitude P plus noise V_N, so it may be represented as (4.1-13), and if $P \gg$ rms V_N, the distribution of R is approximately normal. If, in addition, $P - B \gg$ rms $V_N > 0$, (4.3-15), (4.3-16), and (3.10-19) lead to the approximations

$$I_{dc} = -\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{P} + \frac{1}{\pi}\sqrt{P^2 - B^2} + \frac{\psi_0}{2\pi\sqrt{P^2 - B^2}}$$

$$\approx -\frac{B}{2} + \frac{P}{\pi} + \frac{B^2 + \psi_0}{2\pi P} \tag{4.3-17}$$

$$\overline{I_{lf}^2} \approx \frac{P^2 - B^2}{\pi^2 P^2}\psi_0$$

The second expression for I_{dc} assumes $P \gg B$. When $B = 0$, these reduce to the first terms of (4.2-5) and (4.2-7). By using a different method Middleton has obtained a more precise form of this result.

Incidentally, for a given applied voltage, $I_{dc}(+)$ for a positive bias $|B|$ is related to $I_{dc}(-)$ for a negative bias $-|B|$ by

$$I_{dc}(-) = |B| + I_{dc}(+) \tag{4.3-18}$$

Also r.m.s. $I_{lf}(+)$ is equal to r.m.s. $I_{lf}(-)$. Equation (4.3-18) follows from a physical argument based on the areas underneath a curve of I for the two cases. Both of the above relations follow from formulas given by Middleton when V is the sum of a sine wave plus noise. They may also be derived from (4.3-15) and (4.3-16).

4.4 Output Power Spectrum

The remainder of Part IV will be concerned with methods of solving the following problem: Given a non-linear device and an input voltage consisting of noise alone or of a signal plus noise. What is the power spectrum of the output?

In some ways the answer to this problem gives us less information than the methods discussed in the first three sections. For example, beyond giving the rms value, it tells us very little about the probability density of the current corresponding to a given frequency band of the output. On the other hand, this rms value may be found (by integrating the power spectrum) for any band we choose to study. The methods described earlier depended on the input being confined to a relatively narrow band and gave information regarding only the entire band corresponding to a given harmonic (0th, 1st, 2nd, etc.) of the input. There was no way to study the output when part of a band was eliminated by filters except by obtaining the power spectrum of some function of the envelope.

At present there appear to be two general methods available for the determination of the output power spectrum each with its own advantages and disadvantages. First there is the direct method which has been used by W. R. Bennett*, F. C. Williams**, J. R. Ragazzini[46] and others. The noise is represented as the sum of a finite number of sinusoidal components. The typical modulation product is computed and the output power spectrum is obtained by considering the density and amplitude of these products. The chief advantage of this method lies in its close relation to the known theory of modulation in non-linear circuits. Generally, the lower order modulation products are the only ones which contribute significantly to the output power and when they are known, the problem is well along towards solution. The main disadvantage is the labor of counting the modulation products falling in a given interval. However, Bennett has developed a method for doing this.[47]

The fundamental idea of the second method is to obtain the correlation function for the output current. From this the output power spectrum may be obtained by Fourier's transform. The correlation function method and its variations are of more recent origin than the direct method. They have

* Cited in Section 4.0. Also much of this writer's work on interference in broad band communication systems may be carried over to noise theory without any change in the methods used.

** Cited in Section 4.1.

[46] *Proc. I.R.E.* Vol. 30, pp. 277–288 (June 1942), "The Effect of Fluctuation Voltages on the Linear Detector."

[47] *B.S.T.J.*, Vol. 19 (1940), pp. 587–610, Appendix B.

been discovered independently and at about the same time, by several workers. In a paper read before the I.R.E., Jan. 28, 1944, D. O. North described results obtained by using the correlation function. J. H. Van Vleck and D. Middleton have been using the two variations of the method which we shall describe in Sections 4.7 and 4.8, since early in 1943. A primitive form of the method of Section 4.8 had been used by A. D. Fowler and the writer in some unpublished material written in 1942. Recently, I have learned that a method similar to the one used by Fowler and myself had already been used by Kurt Fränz in 1941.[48]

The correlation function method avoids the problem of counting the modulation products. However, in some cases it becomes rather unwieldy. Probably it is best to have both methods in mind when investigating any particular problem. The direct method will be illustrated by applying it to the square law detector. Two approaches to the correlation function method will then be described and applied to examples.

4.5 Noise Through Square Law-Device

Probably the most direct method of obtaining the power spectrum $W(f)$ of I, where

$$I = \alpha V^2, \tag{4.1-1}$$

V being a noise voltage, is to square the expression

$$V = V_N = \sum_1^M c_m \cos(\omega_m t - \varphi_m) \tag{2.8-6}$$

in which c_m^2 is $2w(f_m)\Delta f$, $\omega_m = 2\pi f_m$, $f_m = m\Delta f$ and $\varphi_1, \varphi_2, \cdots \varphi_M$ are random phase angles.

Considerable simplification of the algebra results when we replace the representation (2.8-6) by

$$V_N = \frac{1}{2}\sum_{-\infty}^{\infty} c_m e^{imat - i\varphi_m} \tag{4.5-1}$$

Here we have added a term $c_0/2$ so as to not have any gaps in the summation and have introduced the definitions

$$\begin{aligned} c_{-m} &= c_m \\ \varphi_{-m} &= -\varphi_m \\ a &= 2\pi\Delta f \end{aligned} \tag{4.5-2}$$

[48] "Die Übertragung von Rauschspannung über den linearen Gleichrichter," *Hochfr. u. Elektroakust.*, June 1941. Other articles by Fränz are (I am indebted to Dr. North for the following references) "Beitrage zur Berechnung des Verhaltnisses von Signal spannung zu Rauschspannung am Ausgang von Empfängern", *E.N.T.*, 17, 215, 1940 and 19, 285, 1942. "Die Amplituden von Geräuschspannungen", *E.N.T.*, 19, 166, 1942. The May 1944 (p. 237), issue of the Wireless Engineer contains an abstract of "The Influence of Carrier Waves on the Noise on the Far Side of Amplitude-Limiters and Linear Rectifiers" by Fränz and Vellat, *E.N.T.*, Vol. 20, pp. 183-189 (Aug. 1943).

Squaring (4.5-1) gives the double series

$$V_N^2 = \frac{1}{4} \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} c_m c_n e^{i(m+n)at - i\varphi_m - i\varphi_n}$$

$$= \frac{1}{4} \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{k-n} c_n e^{ikat - i\varphi_{k-n} - i\varphi_n}$$

Suppose we wish to consider the component of V_N^2 of frequency $f_k = k\Delta f$. It is seen to be

$$A_k \cos(\omega_k t - \psi_k) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} c_{k-n} c_n \cos(kat - \varphi_{k-n} - \varphi_n) \qquad (4.5\text{-}3)$$

The power spectrum $W(f)$ of I at frequency f_k is α^2 times the coefficient of Δf in the mean square value of (4.5-3) where the average is taken over the φ's. Thus

$$W(f_k)\Delta f = \frac{\alpha^2}{4} \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} c_{k-n} c_n c_{k-m} c_m$$

$$\times \text{ ave. } \cos(kat - \varphi_{k-n} - \varphi_n) \cos(kat - \varphi_{k-m} - \varphi_m)$$

where the summations extend over m and n. Let n be fixed and consider those values of m which give an average different from zero. We see that $m = n$ and $m = k - n$ are two such values. The only other possibilities are $m = -n$ and $m = -k + n$, but these lead to terms containing (except when n or k equal zero) three different angles, φ_n, φ_{k-n}, and φ_{k+n} which average to zero. Using the fact that the average of cosine squared is one-half and that for a given n there are two such terms, we get

$$W(f_k)\Delta f = \frac{\alpha^2}{4} \sum_{n=-\infty}^{+\infty} c_{k-n}^2 c_n^2$$

$$= \alpha^2 \Delta f \sum_{n=-\infty}^{+\infty} w(f_k - f_n) w(f_n) \Delta f \qquad (4.4\text{-}5)$$

where in the last step we have used

$$f_{k-n} = (k - n)\Delta f = f_k - f_n$$

and have implied, from $c_{-n} = c_n$, that

$$w(f_{-n}) = w(-n\Delta f) = w(-f_n)$$

is equal to $w(f_n)$.

Thus, from (4.5-4), we get for the power spectrum of I

$$W(f) = \alpha^2 \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \qquad (4.5\text{-}5)$$

with the understanding that f is not zero and

$$w(-x) = w(x). \tag{4.5-6}$$

The result which is obtained by using (2.8-6), involving the cosines and only positive values of m, is

$$W(f) = \alpha^2 \int_0^f w(x)w(f - x)\,dx + 2\alpha^2 \int_0^\infty w(x)w(f + x)\,dx \tag{4.5-7}$$

This contains only positive values of frequency. (4.5-5) and (4.5-7) are equivalent and may readily be transformed into each other.

The first integral in (4.5-7) arises from second order modulation products of the sum type and the second integral from products of the difference type. This may be seen by writing the current as

$$\begin{aligned} I = \alpha V^2 &= \alpha \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \cos(\omega_m t - \varphi_m) \cos(\omega_n t - \varphi_n) \\ &= \frac{\alpha}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \{\cos[(\omega_m - \omega_n)t - \varphi_m + \varphi_n] \\ &\qquad + \cos[(\omega_m + \omega_n)t + \varphi_m + \varphi_n]\} \end{aligned} \tag{4.5-8}$$

The power in the range f_k, $f_k + \Delta f$ is the power due to modulation products of the difference type, $\omega_{k+\ell} - \omega_\ell$, plus the power due to the modulation products of the sum type, $\omega_{k-\ell} + \omega_\ell$. In the first type ℓ runs from 1 to ∞ and in the second type ℓ runs from 1 to $k - 1$.

Consider the difference type first, and for the moment take both k and ℓ to be fixed. The two sets $m = k + \ell$, $n = \ell$ and $m = \ell$, $n = k + \ell$ are the only values of m and n in (4.5-8) leading to $\omega_{k+\ell} - \omega_\ell$. The two corresponding terms in (4.5-8) are equal because $\cos(-x)$ is equal to $\cos x$. The average power contributed by these two terms is

$$\begin{aligned} &\left(\frac{\alpha}{2} c_{k+\ell} c_\ell\right)^2 \times \{\text{Average of } (2 \cos[(\omega_{k+\ell} - \omega_\ell)t - \varphi_{k+\ell} + \varphi_\ell])^2\} \\ &\qquad = \tfrac{1}{2}(\alpha c_{k+\ell} c_\ell)^2 \end{aligned} \tag{4.5-9}$$

The power contributed to f_k, $f_k + \Delta f$ by the difference modulation products is obtained by summing ℓ from 1 to ∞:

$$\begin{aligned} \frac{\alpha^2}{2} \sum_{\ell=1}^{\infty} c_{k+\ell}^2 c_\ell^2 &= 2\alpha^2 \sum_{\ell=1}^{\infty} w(f_{k+\ell}) w(f_\ell) (\Delta f)^2 \\ &\rightarrow 2\alpha^2 \Delta f \int_0^\infty w(f_k + f) w(f)\,df \end{aligned}$$

This leads to the second term in (4.5-7).

Now consider the modulation products of the sum type. The terms of this type in (4.5–8) which give rise to the frequency ω_k are those for which $m + n$ is equal to k. Let n be 1 then $m = k - 1$. The phase of this term is random with respect to all the other terms except the one given by $n = k - 1$, $m = 1$ which has the same phase. The average power contributed by these two terms in (4.5–8) is, as in (4.5–9),

$$\tfrac{1}{2}(\alpha c_1 c_{k-1})^2$$

This disposes of two terms for which $m + n$ is equal to k. Taking n to be 2 and going through the same process gives two more. Thus, assuming for the moment that k is an odd number, the power contributed to the interval $f_k, f_k + \Delta f$ by the sum modulation products is

$$\frac{1}{2} \sum_{n=1}^{(k-1)/2} (\alpha c_n c_{k-n})^2 = \frac{1}{4} \sum_{n=1}^{k-1} (\alpha c_n c_{k-n})^2 \rightarrow \alpha^2 \Delta f \int_0^{f_k} w(f) w(f_k - f)\, df$$

and this leads to the second term in (4.5–7).

When the voltage V applied to the square law device is the sum of a noise voltage V_N and a sine wave:

$$V = P \cos pt + V_N, \tag{4.1–13}$$

we have

$$V^2 = P^2 \cos^2 pt + 2PV_N \cos pt + V_N^2 \tag{4.5–10}$$

From the two equations

$$\cos^2 pt = \frac{1}{2} + \frac{1}{2} \cos 2pt$$

$$\text{ave. } V_N^2 = \sum_1^M c_m^2 \frac{1}{2} \rightarrow \int_0^\infty w(f)\, df$$

we see that I, or αV^2, has a dc component of

$$\frac{\alpha P^2}{2} + \alpha \int_0^\infty w(f)\, df \tag{4.5–11}$$

which agrees with (4.1–14), and a sinusoidal component

$$\frac{\alpha P^2}{2} \cos 2pt \tag{4.5–12}$$

The continuous power spectrum $W_c(f)$ of the remaining portion of I may be computed from

$$2PV_N \cos pt + V_N^2.$$

Using the representation (2.8–6) we see

$$2PV_N \cos pt = P \sum_1^M c_m[\cos(\omega_m t + pt - \varphi_m) + \cos(\omega_m t - pt - \varphi_m)]$$

For the moment, we take $p = 2\pi r\Delta f$. The terms pertaining to frequency $f_n = n\Delta f$ are those for which

$$\begin{aligned} \omega_m + p &= 2\pi f_n & \qquad |\omega_m - p| &= 2\pi f_n \\ m + r &= n & |m - r| &= n \\ m &= n - r & m &= r \pm n \end{aligned}$$

where only positive values of m are to be taken: If $n > r$, then m is $n - r$ or $r + n$. If $n < r$, then m is $r - n$ or $r + n$. In either case the values of m are $|n - r|$ and $n + r$. The terms of frequency f_n in $2PV_N \cos pt$ are therefore

$$Pc_{|n-r|} \cos(2\pi f_n t - \varphi_{|n-r|}) + Pc_{n+r} \cos(2\pi f_n t - \varphi_{n+r})$$

and the mean square value of this expression, the average being taken over the φ's, is

$$\begin{aligned} \frac{P^2}{2}(c_{|n-r|}^2 + c_{n+r}^2) &= P^2 \Delta f[w(f_{|n-r|}) + w(f_{n+r})] \\ &= P^2 \Delta f[w(|f_n - f_p|) + w(f_n + f_p)] \end{aligned}$$

where f_p denotes $p/2\pi$.

By combining this with the expression (4.5–5) which arises from V_N^2 we see that the continuous portion $W_c(f)$ of the power spectrum of I is

$$\begin{aligned} W_c(f) = \alpha^2 P^2[w(f - f_p) + w(f + f_p)] \\ + \alpha^2 \int_{-\infty}^{+\infty} w(x)w(f - x)\,dx \end{aligned} \tag{4.5–13}$$

where $w(-f)$ has the same value as $w(f)$.

Equation (4.5–13) has been used to compute $W_c(f)$ as shown in Fig. 8. The input noise is assumed to be uniform over a band of width β centered at f_p, cf. Filter c, Appendix C. By noting the area under the low frequency portion of the spectrum we find

$$\int_0^\beta W_c(f)\,df = \alpha^2 \beta w_0(P^2 + \beta w_0)$$

Since the mean square value of the input V_N is $\psi_0 = \beta w_0$, it is seen that this equation agrees with the expression (4.1–15) for the mean square value of I_{lf}, the low frequency current, excluding the d.c. If audio frequency filters cut out part of the spectrum, $W_c(f)$ may be integrated over the remaining portion to give the mean square value of the corresponding output current. This idea is mentioned in the footnote pertaining to equation (4.1-6).

If V consists of V_N plus two sinusoidal voltages of incommensurable frequencies, say

$$V = P \cos pt + Q \cos qt + V_N,$$

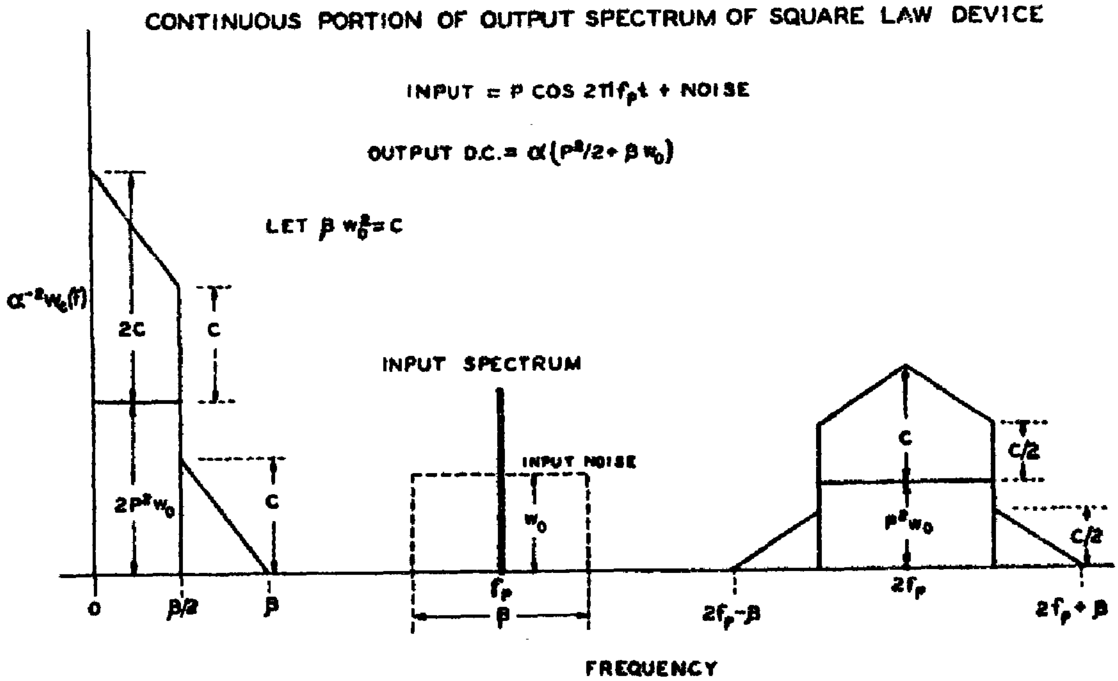

Fig. 8

the continuous portion $W_c(f)$ of the power spectrum of I may be shown to be (4.5-13) plus the additional terms

$$\alpha^2 Q^2[w(f - f_q) + w(f + f_q)] \tag{4.5-14}$$

where f_q denotes $q/2\pi$.

When the voltage applied to the square law device (4.1-1) is[49]

$$V(t) = Q(1 + k \cos pt) \cos qt + V_N$$

$$= Q \cos qt + \frac{Qk}{2} \cos (p + q)t + \frac{Qk}{2} \cos (p - q)t + V_N$$

the resulting current contains the dc component

$$\frac{\alpha}{2} Q^2 \left(1 + \frac{k^2}{2}\right) + \alpha \int_0^\infty w(f)\, df \tag{4.5-16}$$

[49] A complete discussion of this problem is given by L. A. MacColl in a manuscript being prepared for publication.

The sinusoidal terms of I are obtained by squaring

$$Q(1 + k \cos pt) \cos qt$$

and multiplying by α. The remaining portion of I has a continuous power spectrum given by

$$\begin{aligned} W_c(f) = \alpha^2 Q^2 \Big[& w(f - f_q) + w(f + f_q) \\ & + \frac{k^2}{4} w(f - f_p - f_q) + \frac{k^2}{4} w(f + f_p + f_q) \\ & + \frac{k^2}{4} w(f - f_p + f_q) + \frac{k^2}{4} w(f + f_p - f_q) \Big] \\ & + \alpha^2 \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \end{aligned} \tag{4.5-17}$$

where f_p denotes $p/2\pi$ and f_q denotes $q/2\pi$.

4.6 Two Correlation Function Methods

As mentioned in Section 4.4 these methods for determining the output power spectrum are based on finding the correlation function $\Psi(\tau)$ for the output current. From this the power spectrum, $W(f)$, of the output current may be obtained from (2.1-5), rewritten as

$$W(f) = 4 \int_0^\infty \Psi(\tau) \cos 2\pi f \tau \, d\tau \tag{4.6-1}$$

It will be recalled that $W(f)\Delta f$ may be regarded as the average power which would be dissipated by those components of I in the band $f, f + \Delta f$ if I were to flow through a resistance of one ohm.

The input of the non-linear device is taken to be a voltage $V(t)$. It may, for example, consist of a noise voltage $V_N(t)$ plus sinusoidal components. The output is taken to be a current $I(t)$. The non-linear device is specified by a relation between $V(t)$ and $I(t)$. In this work $I(t)$ at time t is assumed to be completely determined by the value of $V(t)$ at time t.

Two methods of obtaining $\Psi(\tau)$ will be described.

(a) Integrating the two-dimensional probability density of $V(t)$ and $V(t + \tau)$ over the values allowed by the non-linear device. This method, which is especially direct when applied to noise alone through rectifiers, was discovered independently by Van Vleck and North.

(b) Introducing and using the characteristic function, which for the sake of brevity will be abbreviated to ch. f., of the two-dimensional probability distribution of $V(t)$ and $V(t + \tau)$.

4.7 Linear Detection of Noise—The Van Vleck-North Method

The method due to Van Vleck and North will be illustrated by using it to determine the output power spectrum of a linear detector when the input consists of noise alone.

The linear detector is specified by

$$I(t) = \begin{cases} 0, & V(t) < 0 \\ V(t), & V(t) > 0, \end{cases} \tag{4.7-1}$$

which may be obtained from (4.2-1) by setting α equal to one, and the input voltage is

$$V(t) = V_N(t) \tag{4.7-2}$$

where $V_N(t)$ is a noise voltage whose correlation function is $\psi(\tau)$ and whose power spectrum is $w(f)$.

The correlation function $\Psi(\tau)$ is the average value of $I(t)I(t+\tau)$. This is the same as the average value of the function

$$F(V_1, V_2) = \begin{cases} V_1 V_2, & \text{when both } V_1, V_2 > 0 \\ 0, & \text{all other } V\text{'s}, \end{cases} \tag{4.7-3}$$

where we have set

$$V_1 = V(t)$$
$$V_2 = V(t+\tau)$$

The two-dimensional distribution of V_1 and V_2 is given by (3.2-4), and from this it follows that the average value of any function $F(V_1, V_2)$ is

$$\int_{-\infty}^{+\infty} dV_1 \int_{-\infty}^{+\infty} dV_2 \frac{F(V_1, V_2)}{2\pi |M|^{1/2}} \exp\left[-\frac{1}{2|M|}(\psi_0 V_1^2 + \psi_0 V_2^2 - 2\psi_\tau V_1 V_2)\right] \tag{4.7-4}$$

where

$$|M| = \psi_0^2 - \psi_\tau^2.$$

For the linear rectifier case, where $F(V_1, V_2)$ is given by (4.7-3), the integral is

$$|M|^{-1/2} \frac{1}{2\pi} \int_0^\infty dV_1 \int_0^\infty dV_2 V_1 V_2 \exp\left[-\frac{1}{2|M|}(\psi_0 V_1^2 + \psi_0 V_2^2 - 2\psi_\tau V_1 V_2)\right]$$
$$= \frac{1}{2\pi}\left([\psi_0^2 - \psi_\tau^2]^{1/2} + \psi_\tau \cos^{-1}\left[\frac{-\psi_\tau}{\psi_0}\right]\right)$$

where we have used (3.5-4) to evaluate the integral. The arc cosine is taken to be between 0 and π. We therefore have for the correlation function of $I(t)$,

$$\Psi(\tau) = \frac{1}{2\pi}\left([\psi_0^2 - \psi_\tau^2]^{1/2} + \psi_\tau \cos^{-1}\left[\frac{-\psi_\tau}{\psi_0}\right]\right) \tag{4.7-5}$$

The power spectrum $W(f)$ may be obtained from this by use of (4.6-1). For this purpose it is convenient to write (4.7-5) in terms of a hypergeometric function. By expanding and comparing terms it is seen that

$$\begin{aligned}\Psi(\tau) &= \frac{\psi_\tau}{4} + \frac{\psi_0}{2\pi} F\left(-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \frac{\psi_\tau^2}{\psi_0^2}\right) \\ &= \frac{\psi_\tau}{4} + \frac{\psi_0}{2\pi} + \frac{\psi_\tau^2}{4\pi\psi_0} + \text{terms involving } \psi_\tau^4, \psi_\tau^6, \text{etc.}\end{aligned} \tag{4.7-6}$$

As will be discussed more fully in Section 4.8, a constant term A^2 in $\psi(\tau)$ indicates a direct current component of $I(t)$ of A amperes. Thus $I(t)$ has a *dc* component equal to

$$\left[\frac{\psi_0}{2\pi}\right]^{1/2} = \frac{1}{\sqrt{2\pi}} \times \text{rms value of } V(t) \tag{4.7-7}$$

This agrees with (4.2-3) when the P of that equation is set equal to zero.

Integrals of the form

$$G_n(f) = \int_0^\infty \psi_\tau^n \cos 2\pi f\tau \, d\tau$$

which result when (4.7-6) is put in (4.6-1) and integrated termwise are discussed in Appendix 4C. From the results given there it is seen that if we neglect ψ_τ^4 and higher powers we obtain an approximation for the continuous portion $W_c(f)$ of $W(f)$:

$$\begin{aligned}W_c(f) &\approx G_1(f) + \frac{G_2(f)}{\pi\psi_0} \\ &= \frac{w(f)}{4} + \frac{1}{4\pi\psi_0}\frac{1}{2}\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx\end{aligned} \tag{4.7-8}$$

where $w(-f)$ is defined as $w(f)$.

When $V_N(t)$ is uniform over a relatively narrow band extending from f_a to f_b so that $w(f)$ is equal to w_0 in this band and is zero outside it, we may use the results for Filter c of Appendix 4C. The f_0 and β given there are related to f_a and f_b by

$$f_a = f_0 - \frac{\beta}{2}, \qquad f_b = f_0 + \frac{\beta}{2}$$

and the value of w_0 taken there is the same as here and is ψ_0/β. The value of $G_2(f)$ given there leads to the approximation, for low frequencies:

$$\begin{aligned} W_o(f) &= \frac{1}{\pi\psi_0}\frac{\psi_0^2}{4\beta}\left(1-\frac{f}{\beta}\right) \\ &= \frac{w_0}{4\pi}\left(1-\frac{f}{f_b-f_a}\right) \end{aligned} \tag{4.7-9}$$

when $0 < f < f_b - f_a$, and to $W_0(f) = 0$ for $f_b - f_a < f < f_a$. By setting P equal to zero in the curve given in Fig. 8 for $W_c(f)$ corresponding to the square law detector, we see that the low frequency portion of the power spectrum is triangular in shape and is zero at $f = \beta$. Thus, looking at (4.7-9), we see that to a first approximation the shape of the output power spectrum is the same for a linear detector as for a square law detector when the input consists of a relatively narrow band of noise.

An approximate rms value of the low frequency output current may be obtained by integrating (4.7-9)

$$\begin{aligned} \overline{I_{lf}^2} &= \int_0^{f_b-f_a} W_o(f)\,df \\ &\approx \frac{w_0(f_b-f_a)}{8\pi} = \frac{\psi_0}{8\pi} \end{aligned}$$

$$\text{rms low freq. current} = \frac{1}{\sqrt{8\pi}} \times \text{rms applied voltage} \tag{4.7-10}$$

It is seen that this is half of the direct current. It must be kept in mind that (4.7-10) is an approximation because we have neglected ψ_r^4 and higher powers. The true value may be obtained from (4.2-8). It is seen that the coefficient $(8\pi)^{-1/2} = 0.200$ should be replaced by

$$\frac{1}{\pi}\left(2-\frac{\pi}{2}\right)^{1/2} = 0.209$$

$W_c(f)$ for other types of band pass filters may be obtained by using the corresponding G's given in appendix 4C. It turns out that (4.7-10) holds for all three types of filters. This is a special case of Middleton's theorem, mentioned several times before, that the total power in any modulation product (it will be shown later in Section 4.9 that the term ψ_r^n in (4.7-6) corresponds to the n^{th} order modulation products) depends only on the total input power of the applied noise, not on its spectral distribution.

4.8 The Characteristic Function Method

As mentioned in the preceding parts, especially in connection with equation (1.4-3), the ch. f. of a random variable x is the average value of exp

(iux). This is a function of u. The ch. f. of two random variables x and y is the average value of exp $(iux + ivy)$ and is a function of u and v. The ch. f. which we shall use here is the ch. f. of the two random variables $V(t)$ and $V(t + \tau)$ where $V(t)$ is the voltage applied to the non-linear device, and the randomness is introduced by t being selected at random, τ remaining fixed. We may write this characteristic function as

$$g(u, v, \tau) = \operatorname*{Limit}_{T \to \infty} \frac{1}{T} \int_0^T \exp [iuV(t) + ivV(t + \tau)] \, dt \quad (4.8\text{–}1)$$

If $V(t)$ contains a noise voltage $V_N(t)$, as it always does in this section, and if we use the representation (2.8–1) or (2.8–6) a large number of random parameters (a_n's and b_n's or φ_n's) will appear in (4.8–1). In accordance with our use of such representations we may average over these parameters without changing the value of (4.8–1) and may thereby simplify the integration.

For example suppose

$$V(t) = V_s(t) + V_N(t) \quad (4.8\text{–}2)$$

where $V_s(t)$ is some regular voltage which may, e.g., consist of one or more sine waves. Substituting this in (4.8–1) and using the result (3.2–7) that the ch. f. of $V_N(t)$ and $V_N(t + \tau)$ is

$$\begin{aligned} g_N(u, v, \tau) &= \text{ave. } \exp [iuV_N(t) + ivV_N(t + \tau)] \\ &= \exp \left[-\frac{\psi_0}{2} (u^2 + v^2) - \psi_\tau uv \right] \end{aligned} \quad (4.8\text{–}3)$$

$\psi_\tau \equiv \psi(\tau)$ being the correlation function of $V_N(t)$, we obtain for the ch. f. of $V(t)$ and $V(t + \tau)$,

$$\begin{aligned} g(u, v, \tau) &= \exp \left[-\frac{\psi_0}{2} (u^2 + v^2) - \psi_\tau uv \right] \\ &\quad \times \operatorname*{Limit}_{T \to \infty} \frac{1}{T} \int_0^T \exp [iuV_s(t) + ivV_s(t + \tau)] \, dt \\ &= g_N(u, v, \tau) g_s(u, v, \tau) \end{aligned} \quad (4.8\text{–}4)$$

In the last line we have used $g_s(u, v, \tau)$ to denote the limit in the line above:

$$g_s(u, v, \tau) = \operatorname*{Limit}_{T \to \infty} \frac{1}{T} \int_0^T \exp [iuV_s(t) + ivV_s(t + \tau)] \, dt \quad (4.8\text{–}5)$$

The principal reason we use the ch. f. is because quite a few non-linear devices may be described by the integral

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu} \, du \quad (4A\text{–}1)$$

where the function $F(iu)$ and the path of integration C are chosen to fit the device. Examples of such devices are given in Appendix 4A. The correlation function $\Psi(\tau)$ of $I(t)$ is given by

$$\begin{aligned}\Psi(\tau) &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt \\ &= \operatorname*{Limit}_{T\to\infty} \frac{1}{4\pi^2 T}\int_0^T dt \int_C F(iu)e^{iuV(t)}\,du \int_C F(iv)e^{ivV(t+\tau)}\,dv \\ &= \frac{1}{4\pi^2}\int_C F(iu)\,du \int_C F(iv)\,dv \\ &\qquad \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T \exp\,[iuV(t) + ivV(t+\tau)]\,dt \\ &= \frac{1}{4\pi^2}\int_C F(iu)\,du \int_C F(iv)g(u, v, \tau)\,dv\end{aligned} \tag{4.8-6}$$

This is the fundamental formula of the ch. f. method.

When $V(t)$ is the sum of a noise voltage and a regular voltage, as in (4.8-2), (4.8-6) becomes

$$\Psi(\tau) = \frac{1}{4\pi^2}\int_C F(iu)e^{-(\psi_0/2)u^2}\,du \int_C F(iv)e^{-(\psi_0/2)v^2} e^{-\psi_\tau uv} g_s(u, v, \tau)\,dv \tag{4.8-7}$$

where $g_s(u, v, \tau)$ is the ch. f. of $V_s(t)$ and $V_s(t+\tau)$ given by (4.8-5). This is a definite expression for $\Psi(\tau)$. All that follows is devoted to the evaluation of this integral and to the evaluation of

$$W(f) = 4\int_0^\infty \Psi(\tau)\cos 2\pi f\tau\,d\tau \tag{4.6-1}$$

for the power spectrum of I.

Quite often $I(t)$ will contain dc and periodic components. It seems convenient to deal with these separately since they correspond to terms in $\Psi(\tau)$ which cause the integral (4.6-1) for $W(f)$ to diverge. In fact, from Section 2.2 it follows that a correlation function of the form

$$A^2 + \frac{C^2}{2}\cos 2\pi f_0\tau \tag{2.2-3}$$

corresponds to a current

$$A + C\cos\,(2\pi f_0 t - \varphi) \tag{2.2-2}$$

where the phase angle φ cannot be determined from (2.2–3) since it does not affect the average power.

Consider the correlation function for $V(t) = V_s(t) + V_N(t)$ given by (4.8–2). It is

$$\begin{aligned}\operatorname*{Limit}_{T\to\infty} \frac{1}{T}\Bigg[\int_0^T V_s(t)V_s(t+\tau)\,dt + \int_0^T V_s(t)V_N(t+\tau)\,dt \\ + \int_0^T V_N(t)V_s(t+\tau)\,dt + \int_0^T V_N(t)V_N(t+\tau)\,dt\Bigg]\end{aligned} \tag{4.8–8}$$

Since $V_s(t)$ and $V_N(t)$ are unrelated the contributions of the second and third integrals vanish leaving us with the result

$$\begin{aligned}\text{Correlation function of } V(t) = \text{Correlation function of } V_s(t) \\ + \text{Correlation function of } V_N(t).\end{aligned} \tag{4.8–9}$$

Now as $\tau \to \infty$ the correlation function of $V_N(t)$ becomes zero while that of $V_s(t)$ becomes of the type (2.2–3) given above. Hence the correlation function of the regular voltage $V_s(t)$ may be obtained from $V(t)$ by letting $\tau \to \infty$ and picking out the non-vanishing terms. Although we have been speaking of $V(t)$, the same results hold for $I(t)$ and this process may be used to pick out those parts of $\Psi(\tau)$ which correspond to the *dc* and periodic components of $I(t)$. Thus, if we look at (4.8–7) we see that as $\tau \to \infty, \psi_\tau \to 0$, while the $g_s(u, v, \tau)$ corresponding to $V_s(t)$ given by (4.8–5) remains unchanged in general magnitude. This last statement may be hard to see, but examination of the cases discussed later show that it is true, at least for these cases. Thus the portion of $\Psi(\tau)$ corresponding to the *dc* and periodic components of $I(t)$ is, setting $\psi_\tau = 0$ in (4.8–7),

$$\Psi_\infty(\tau) = \frac{1}{4\pi^2}\int_C F(iu)e^{-(\psi_0/2)u^2}\,du \int_C F(iv)e^{-(\psi_0/2)v^2} g_s(u, v, \tau)\,dv \tag{4.8–10}$$

where the subscript ∞ indicates that $\Psi_\infty(\tau)$ is that part of $\Psi(\tau)$ which does not vanish as $\tau \to \infty$.

We may write (4.8–9), when applied to $I(t)$, as

$$\Psi(\tau) = \Psi_\infty(\tau) + \Psi_c(\tau) \tag{4.8–11}$$

where $\Psi_c(\tau)$ is the correlation function of the "continuous" portion of the power spectrum of $I(t)$.

Incidentally, the separation of $\Psi(\tau)$ into the two parts shown in (4.8–11) may be avoided if one is willing to use the $\delta(f)$ functions in order to interpret the integral in (4.6–1) as explained in Section 2.2. This method gives the proper *dc* and sinusoidal components even though (4.6–1) does not converge (because of the presence of the terms leading to $\Psi_\infty(\tau)$).

4.9 NOISE PLUS SINE WAVE APPLIED TO NON-LINEAR DEVICE

In order to illustrate the characteristic function method described in Section 4.8 we shall consider the case of a non-linear device specified by

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu} \, du \tag{4A-1}$$

when V consists of a noise voltage plus a sine wave:

$$V(t) = P \cos pt + V_N(t) \tag{4.1-13}$$

As usual, $V_N(t)$ has the power spectrum $w(f)$ and the correlation function $\psi(\tau)$. $\psi(\tau)$ is often written as ψ_τ for the sake of shortness. Comparing (4.1-13) with (4.8-2) gives

$$V_s(t) = P \cos pt \tag{4.9-1}$$

Our first task is to compute the ch. f. $g_s(u, v, \tau)$ for the pair of random variables $V_s(t)$ and $V_s(t + \tau)$. We do this by using the integral (4.8-5):

$$\begin{aligned} g_s(u, v, \tau) &= \underset{T \to \infty}{\text{Limit}} \frac{1}{T} \int_0^T \exp\,[iuP \cos pt + ivP \cos p(t + \tau)]\, dt \\ &= J_0(P\sqrt{u^2 + v^2 + 2uv \cos p\tau}) \end{aligned} \tag{4.9-2}$$

where J_0 is a Bessel function. The integration is performed by writing

$$\begin{aligned} u \cos pt + v \cos p\,(t + \tau) &= (u + v \cos p\tau) \cos pt - v \sin p\tau \sin pt \\ &= \sqrt{u^2 + v^2 + 2uv \cos p\tau}\, \cos\,(pt + \text{phase angle}) \end{aligned}$$

and using the integral

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos t} \, dt$$

The correlation function for (4.1-13) has also been given in Section 3.10.

The correlation function $\Psi(\tau)$ for $I(t)$ may now be obtained by substituting the above expressions in (4.8-7)

$$\begin{aligned} \Psi(\tau) = \frac{1}{4\pi^2} \int_C du\, F(iu) e^{-(\psi_0/2)u^2} \int_C dv\, F(iv) e^{-(\psi_0/2)v^2} \\ e^{-\psi_\tau uv} J_0(P\sqrt{u^2 + v^2 + 2uv \cos p\tau}). \end{aligned} \tag{4.9-3}$$

$\Psi_\infty(\tau)$, the correlation function for the d.c. and periodic components of I, may, according to (4.8-10), be obtained from this by setting ψ_τ equal to zero.

When we have a particular non-linear device in mind the appropriate $F(iu)$ may often be obtained from Appendix 4A. For example, $F(iu)$ for a linear rectifier is $-u^{-2}$. Inserting this value in (4.9-3) gives a definite double integral for $\Psi(\tau)$. If there were some easy way to evaluate this integral then everything would be fine. Unfortunately, no simple method of evaluation has yet been found. However, one method is available which is closely related to the direct method used by Bennett. It is based on the expansion

$$g_s(u, v, \tau) = J_0(P\sqrt{u^2 + v^2 + 2uv\cos p\tau})$$

$$= \sum_{n=0}^{\infty} \epsilon_n(-)^n J_n(Pu)J_n(Pv) \cos np\tau \qquad (4.9\text{–}4)$$

$$\epsilon_0 = 1, \quad \epsilon_n = 2 \quad \text{for} \quad n \geq 1$$

This expansion enables us to write the troublesome terms in (4.9–3) as

$$e^{-\psi_\tau uv} J_0(P\sqrt{u^2 + v^2 + 2uv\cos p\tau})$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-)^{n+k} \epsilon_n \cos np\tau \frac{(\psi_\tau uv)^k}{k!} J_n(Pu)J_n(Pv) \qquad (4.9\text{–}5)$$

The virtue of this double sum is that it simplifies the integration. Thus, putting it in (4.9–3) and setting

$$h_{nk} = \frac{i^{n+k}}{2\pi} \int_C F(iu)u^k J_n(Pu)e^{-(\psi_0/2)u^2}\, du \qquad (4.9\text{–}6)$$

gives

$$\Psi(\tau) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \psi_\tau^k h_{nk}^2 \epsilon_n \cos np\tau \qquad (4.9\text{–}7)$$

The correlation function $\Psi_\infty(\tau)$ for the dc and periodic components of I are obtained by letting $\tau \to \infty$ where $\psi_\tau \to 0$. Only the terms for which $k = 0$ remain:

$$\Psi_\infty(\tau) = \sum_{n=0}^{\infty} \epsilon_n h_{n0}^2 \cos np\tau \qquad (4.9\text{–}8)$$

Comparing this with the known fact that the correlation function of

$$A + C\cos(2\pi f_0 t - \varphi) \qquad (2.2\text{–}2)$$

is

$$A^2 + \frac{C^2}{2}\cos 2\pi f_0 \tau \qquad (2.2\text{–}3)$$

and remembering that ϵ_0 is one while ϵ_n is two for $n \geq 1$ shows that

$$\text{Amplitude of dc component of } I = h_{00}$$

$$\text{Amplitude of } \frac{np}{2\pi} \text{ component of } I = 2h_{n0} \qquad (4.9\text{–}9)$$

Incidentally, these expressions for the amplitudes follow almost at once from the direct method of solution. This will be shown in connection with equation (4.9–17).

Since the correlation function $\Psi_c(\tau)$ for the continuous portion $W_c(f)$ of the power spectrum for I is given by

$$\Psi_c(\tau) = \Psi(\tau) - \Psi_\infty(\tau), \tag{4.8–11}$$

we also have

$$\Psi_c(\tau) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k!} \psi_\tau^k h_{nk}^2 \epsilon_n \cos np\tau \tag{4.9–10}$$

When this is substituted in

$$W_c(f) = 4 \int_0^\infty \Psi_c(\tau) \cos 2\pi f\tau \, d\tau \tag{4.9–11}$$

we obtain

$$W_c(f) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{2\epsilon_n}{k!} h_{nk}^2 \left[G_k\left(f - \frac{np}{2\pi}\right) + G_k\left(f + \frac{np}{2\pi}\right) \right] \tag{4.9–12}$$

where

$$G_k(f) = \int_0^\infty \psi_\tau^k \cos 2\pi f\tau \, d\tau \tag{4.9–13}$$

is the function studied in Appendix 4C. $G_k(f)$ is an even function of f. The double series (4.9–12) for W_c looks rather formidable. However, when we are interested in a particular portion of the frequency spectrum often only a few terms of the series are needed.

It has been mentioned above that the direct method of obtaining the output power spectrum is closely related to the equations just derived. We now study this relation.

We start with the following result from modulation theory[50]: Let the voltage

$$\begin{aligned} V &= P_0 \cos x_0 + P_1 \cos x_1 + \cdots + P_N \cos x_N \\ x_k &= p_k t, \qquad k = 0, 1, \cdots N, \end{aligned} \tag{4.9–14}$$

where the p_k's are incommensurable, be applied to the device (4A–1). The output current is

$$\begin{aligned} I = \sum_{m_0=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \tfrac{1}{2} A_{m_0 \cdots m_N} \epsilon_{m_0} \\ \cdots \epsilon_{m_N} \cos m_0 x_0 \cos m_1 x_1 \cdots \cos m_N x_N \end{aligned} \tag{4.9–15}$$

[50] Bennett and Rice, "Note on Methods of Computing Modulation Products," *Phil. Mag.* S.7, V. 18, pp. 422–424, Sept. 1934, and Bennett's paper cited in Section 4.0.

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. When the product of the cosines is expressed as a sum of cosines of the angles $m_0 x_0 \pm m_1 x_1 \cdots \pm m_N x_N$, it is seen that the coefficient of the typical term is $A_{m_0 \cdots m_N}$, except when all the m's are zero in which case it is $\frac{1}{2} A_{0 \cdots 0}$. Thus

$$\frac{1}{2} A_{00 \cdots 0} = \text{dc component of } I$$

$$| A_{m_0 \cdots m_N} | = \text{amplitude of component of frequency} \qquad (4.9\text{–}16)$$

$$\frac{1}{2\pi} | m_0 p_0 \pm m_1 p_1 \pm \cdots \pm m_N p_N |$$

For all values of the m's,

$$A_{m_0 \cdots m_N} = \frac{i^M}{\pi} \int_C F(iu) \prod_{r=0}^{N} J_{m_r}(P_r u) \, du$$

$$M = m_0 + m_1 + \cdots + m_N \qquad (4.9\text{–}17)$$

Following Bennett's procedure, we identify V as given by (4.9–14), with

$$V = P \cos pt + V_N \qquad (4.1\text{–}13)$$

by setting $P_0 = P$, $p_0 = p$, and representing the noise voltage V_N by the sum of the remaining terms. Since this makes P_1, P_N all very small, Laplace's process indicates that in (4.9–17) we may put

$$\prod_{r=1}^{N} J_0(P_r u) = \exp - \frac{u^2}{4} (P_1^2 + \cdots + P_N^2)$$

$$\approx e^{-\psi_0 u^2/2} \qquad (4.9\text{–}18)$$

We have used the fact that ψ_0 is the mean square value of V_N. It follows from these equations that

$$\text{dc component of } I = \frac{1}{2\pi} \int_C F(iu) J_0(Pu) e^{(-\psi_0/2)u^2} \, du$$

$$\text{Component of frequency } \frac{np}{2\pi} = \frac{i^n}{\pi} \int_C F(iu) J_n(Pu) e^{-\psi_0 u^2/2} \, du$$

These results are identical with those of (4.9–9).

The equations just derived show that h_{n0} is to be associated with the n^{th} harmonic of p. In much the same way it may be shown that h_{nk} is to be associated with the modulation products arising from the n^{th} harmonic of p and k of the elementary sinusoidal components representing V_N. We consider only combinations of the form $p_1 \pm p_2 \pm p_3$, taking $k = 3$ for example, and neglect terms of the form $3p_1$ and $2p_1 \pm p_2$. The former type is much more numerous, there being about N^3 of them while there are only about N and N^2, respectively, of the latter type.

We again take $k = 3$ and consider m_1, m_2, m_3 to be one, and $m_4, \cdots m_N$ to be zero, corresponding to the modulation product $np \pm p_1 \pm p_2 \pm p_3$. By making the same sort of approximations as Bennett does we find

$$A_{n,1,1,1,0,0\cdots0} = \frac{i^{n+3}}{\pi} \frac{P_1 P_2 P_3}{8} \int_C F(iu) J_n(Pu) u^3 e^{(-u^2/2)\psi_0} \, du$$

$$= \frac{P_1 P_2 P_3}{4} h_{n3}$$

When any other modulation product of the form $np \pm p_{r_1} \pm p_{r_2} \pm p_{r_3}$ is considered we get a similar expression in which $P_1P_2P_3$ is replaced by $P_{r_1}P_{r_2}P_{r_3}$. This may be done for any value of k. The result indicates that h_{nk}, and consequently also the $(n, k)^{\text{th}}$ terms in the double series (4.9–10) and (4.9–12) for $\Psi_c(\tau)$ and $W_c(f)$, are to be associated with the modulation products of order (n, k), the n referring to the signal and the k to the noise components.

We now may state a theorem due to Middleton regarding the total power in the modulation products of a given order. For a given non-linear device (i.e. $F(iu)$ is given), the total power which would be dissipated by all of the modulation products which are of order (n, k) if I were to flow through a resistance of one ohm is

$$\Psi_{nk}(0) = \frac{\epsilon_n[\psi(0)]^k}{k!} h_{nk}^2 = \frac{\epsilon_n[\overline{V_N^2}]^k h_{nk}^2}{k!} \tag{4.9–19}$$

The important feature of this expression is that it depends only on the r.m.s. value of V_N and on $F(iu)$. It depends not at all upon the spectral distribution of the noise power in the input.

The proof of (4.9–19) is based on the relation

$$\Psi_{nk}(0) = \int_0^\infty W_{nk}(f) \, df$$

between the total power dissipated by all the (n, k) order products and the corresponding correlation function obtained from (4.9–7).

This theorem has been used by Middleton to show that when the input is confined to a relatively narrow frequency band, so that the output spectrum consists of bands, the power in each band depends only on $\overline{V_N^2}$ and not on the spectrum of V_N.

4.10 Miscellaneous Results Obtained by Correlation Function Method

In this section a number of results which may be obtained from the theory given in the sections following 4.6 are given.

When the input to the square law device

$$I = \alpha V^2 \qquad (4.1\text{–}1)$$

consists of noise only, so that $V = V_N$, the correlation function for I is

$$\Psi(\tau) = \alpha^2[\psi_0^2 + 2\,\psi_\tau^2] \qquad (4.10\text{–}1)$$

where ψ_τ is the correlation function of V_N. This may be compared with equation (3.9–7). When V is general,

$$\begin{aligned}\Psi(\tau) &= \text{ave. } I(t)I(t+\tau)\\ &= \text{ave. } \alpha^2 V^2(t)V^2(t+\tau)\\ &= \alpha^2 \times \text{Coefficient of } \frac{(iu)^2}{2!}\frac{(iv)^2}{2!} \text{ in power series expansion} \qquad (4.10\text{–}2)\\ &\qquad\qquad \text{of ch. f. of } V(t),\, V(t+\tau)\end{aligned}$$

where we have used a known property of the characteristic function. An expression for the ch. f., denoted by $g(u, v, \tau)$, is given by (4.8–4). For example, when V consists of a sine wave plus noise, (4.1–13), the ch. f. is obtainable from (4.9–3). Hence,

$$\begin{aligned}\Psi(\tau) &= \text{Coeff. of } \frac{u^2 v^2}{4} \text{ in expansion of}\\ &\qquad \alpha^2 J_0(P\sqrt{u^2 + v^2 + 2uv\cos p\tau})\\ &\qquad \times \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \qquad (4.10\text{–}3)\\ &= \alpha^2\left[\left(\frac{P}{2} + \psi_0\right)^2 + \frac{P^4}{8}\cos 2p\tau + 2P^2\psi_\tau\cos p\tau + 2\psi_\tau^2\right]\end{aligned}$$

The first two terms give the dc and second harmonic. The last two terms may be used to compute $W_c(f)$ as given by (4.5–13).

Expressions (4.10–1) and (4.10–3) are special cases of results obtained by Middleton who has studied the general theory of the quadratic rectifier by using the Van Vleck-North method, described in Section 4.7.

As an example to which the theory of Section 4.9 may be applied we consider the sine wave plus noise, (4.1–13), to be applied to the ν-law rectifier

$$\begin{aligned} I &= 0, \qquad V < 0\\ I &= V^\nu, \qquad V > 0 \end{aligned} \qquad (4.10\text{–}4)$$

From the table in Appendix 4A it is seen that

$$F(iu) = \Gamma(\nu + 1)(iu)^{-\nu-1}$$

and that the path of integration C runs along the real axis from $-\infty$ to ∞ with a downward indentation at the origin. The integral (4.9-6) for h_{nk} becomes

$$h_{nk} = \frac{i^{n+k-\nu-1}}{2\pi} \Gamma(\nu + 1) \int_C u^{k-\nu-1} J_n(Pu) e^{-(\psi_0/2)u^2} \, du$$

$$= \frac{\left(\frac{\psi_0}{2}\right)^{(\nu-k)/2} x^{n/2} \Gamma(\nu + 1)}{2\Gamma\left(\frac{2 - k - n + \nu}{2}\right) n!} \, {}_1F_1\left(\frac{k + n - \nu}{2}; n + 1; -x\right) \quad (4.10\text{–}5)$$

$$x = \frac{P^2}{2\psi_0}$$

where the integration has been performed by expanding $J_n(Pu)$ in powers of u and using

$$\int_C e^{-au^2} u^{2\lambda-1} \, du = ie^{-\lambda i\pi} a^{-\lambda} \sin \lambda\pi \Gamma(\lambda)$$

$$= \frac{a^{-\lambda}}{2} (1 - e^{-2\lambda i\pi}) \Gamma(\lambda) \quad (4.10\text{–}6)$$

$$= \frac{i\pi e^{-\lambda i\pi}}{a^{\lambda} \Gamma(1 - \lambda)}$$

it being understood that $\arg u = 0$ on the positive portion of C.

From (4.9-9), the *dc* component of I is

$$h_{00} = \frac{\Gamma(1 + \nu)}{2\Gamma\left(1 + \frac{\nu}{2}\right)} \left(\frac{\psi_0}{2}\right)^{\nu/2} {}_1F_1\left(-\frac{\nu}{2}; 1; -x\right) \quad (4.10\text{–}7)$$

which reduces to the expression (4.2-3) when $\nu = 1$ for the linear rectifier (aside from the factor α).

When the input (sine wave plus noise) is confined to a relatively narrow band, and when we are interested in the low frequency output, consideration of the modulation products suggests that we consider the difference products from the products of order (0, 0), (0, 2), (0, 4), $\cdots$ (1, 1), (1, 3), $\cdots$ (2, 0), (2, 2), $\cdots$ etc. where the typical product is of order (n, k). The orders (0, 0) and (2, 0) give the *dc* and second harmonic and hence are not considered in the computation of $W_c(f)$. Of the remaining terms, either (0, 2) or (1, 1) gives the greatest contribution to the series (4.9-12) and (4.9-10) for $W_c(f)$ and $\Psi_c(\tau)$. The remaining terms contribute less and less as n and

k increase. The low frequency portion of the continuous portion of the output power spectrum is then, from (4.9–12),

$$W_o(f) = \frac{4}{2!} h_{02}^2 G_2(f) + \frac{4}{4!} h_{04}^2 G_4(f) + \cdots$$

$$+ \frac{4}{1!} h_{11}^2 [G_1(f - f_0) + G_1(f + f_0)] + \frac{4}{3!} h_{13}^2 [G_3(f - f_0) \quad (4.10\text{–}8)$$

$$+ G_3(f + f_0)] + \frac{4}{2!} h_{22}^2 [G_2(f - 2f_0) + G_2(f + 2f_0)] + \cdots$$

From Table 2 of Appendix 4C we may pick out the low frequency portions of the G's. It must be remembered that $G_m(x)$ is an even function of x and that $0 < f \ll f_0$.

As an example we take the input noise V_N to have the same $w(f)$ and $\psi(\tau)$ as Filter a, the normal law filter, of Appendix 4C, so that

$$w(f) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(f-f_0)^2/2\sigma^2}$$

and assume that the sine wave signal is at the middle of the band, giving $p = 2\pi f_0$. Thus, from (4.10–8), for low frequencies and the normal law distribution of the input noise power,

$$W_o(f) = \frac{1}{4\sigma\sqrt{\pi}} h_{02}^2 \psi_0^2 e^{-f^2/4\sigma^2} + \frac{1}{64\sigma\sqrt{2\pi}} h_{04}^2 \psi_0^4 e^{-f^2/8\sigma^2}$$

$$+ \frac{2}{\sigma\sqrt{2\pi}} h_{11}^2 \psi_0 e^{-f^2/2\sigma^2} + \frac{1}{4\sigma\sqrt{6\pi}} h_{13}^2 \psi_0^3 e^{-f^2/6\sigma^2} \quad (4.10\text{–}9)$$

$$+ \frac{1}{4\sigma\sqrt{\pi}} h_{22}^2 \psi_0^2 e^{-f^2/4\sigma^2} + \cdots$$

Although we have been speaking of the ν-law rectifier, equation (4.10–9) gives the low frequency portion of $W_o(f)$, corresponding to a normal law noise power, for any non-linear device provided the proper h_{nk}'s are inserted.

When we set ν equal to one in the expression (4.10–5) for h_{nk} we may obtain the results given by Bennett. Middleton has studied the output of a biased linear rectifier, when the input consists of a sine wave plus noise, and also the special case of the unbiased linear rectifier. He has computed the output for a wide range of the ratios P^2/ψ_0, B^2/ψ_0 where B is the bias. In order to cover the entire range he had to derive two series for the corresponding h_{nk}'s, each series being suitable for its particular portion of the range.

A special case of (4.10-9) occurs when noise alone is applied to a linear rectifier. The low frequency portion of the output power spectrum is

$$W_o(f) = \frac{\psi_0}{\pi}\sum_{m=1}^{\infty}\frac{(-\frac{1}{2})_m(-\frac{1}{2})_m}{m!\,m!}\,\frac{1}{\sigma\sqrt{4m\pi}}\,e^{-f^2/4m\sigma^2}$$

$$= \frac{\psi_0\pi^{-3/2}}{2\sigma}\left[\tfrac{1}{4}e^{-f^2/4\sigma^2} + \frac{1}{64\sqrt{2}}e^{-f^2/8\sigma^2} + \frac{1}{256\sqrt{3}}e^{-f^2/12\sigma^2} + \cdots\right] \qquad (4.10\text{–}10)$$

where we have used (4.7–6) and Table 2 of Appendix 4C.

The correlation function of

$$V_s = P\cos pt + Q\cos qt,$$

where p and q are incommensurable, is

$$J_0(P\sqrt{u^2+v^2+2uv\cos p\tau}) \times J_0(Q\sqrt{u^2+v^2+2uv\cos q\tau})$$

From equations (4.9–16) and (4.9–17) it is seen immediately that

$$h_{000} = \frac{1}{2\pi}\int_C F(iu)J_0(Pu)J_0(Qu)e^{-(u^2/2)\psi_0}\,du \qquad (4.10\text{–}11)$$

is the d.c. component of I when the applied voltage is

$$P\cos pt + Q\cos qt + V_N. \qquad (4.1\text{–}4)$$

J. R. Ragazzini has obtained an approximate expression for the output power spectrum when the voltage

$$V = V_s + V_N$$
$$V_s = Q(1 + r\cos pt)\cos qt \qquad (4.10\text{–}12)$$

is impressed on a linear rectifier.[46] In terms of our notation his expression for the continuous portion of the power spectrum is (for low frequencies)

$$W_c(f) = \frac{1}{\pi^2\alpha^2(Q^2+2\psi_0)} \times \begin{bmatrix} W_c(f) \text{ given by equation} \\ (4.5\text{–}17) \text{ for square law device}\end{bmatrix} \qquad (4.10\text{–}13)$$

The α^2 is put in the denominator to cancel the α^2 in the expression (4.5–17). We take the linear rectifier to be

$$I = \begin{cases} 0, & V < 0 \\ V, & 0 < V \end{cases} \qquad (4.10\text{–}14)$$

and replace the index of modulation, k, in (4.5–17) by r.

[46] Equation (12), "The Effect of Fluctuation Voltages on the Linear Detector," *Proc. I.R.E.*, V. 30, pp. 277–288 (June 1942).

Ragazzini's formula is quite accurate when the index of modulation r is small, especially when $y = Q^2/(2\psi_0)$ is large. To show this we put $r = 0$ in (4.10–13) and obtain

$$W_0(f) = \frac{1}{\pi^2(Q^2 + 2\psi_0)} \left[Q^2 w(f_q - f) + Q^2 w(f_q + f) + \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \right] \tag{4.10–15}$$

where $f_q = q/(2\pi)$. This is to be compared with the low frequency portion of $W_c(f)$ obtained by specializing (4.10–8) to obtain the output power spectrum of a linear rectifier when the input consists of a sine wave plus noise. The leading terms in (4.10–8) give

$$W_c(f) = h_{11}^2[w(f_q - f) + w(f_q + f)] + h_{02}^2 \frac{1}{4} \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \tag{4.10–16}$$

The values of the h's appropriate to a linear rectifier are obtained by setting $\nu = 1$ in (4.10–5) and noticing that Q now plays the role of P.

$$\begin{aligned} h_{11} &= \frac{1}{2}\left(\frac{y}{\pi}\right)^{1/2} {}_1F_1(\tfrac{1}{2}; 2; -y) \\ h_{02} &= (2\pi\psi_0)^{-1/2} {}_1F_1(\tfrac{1}{2}; 1; -y) \\ y &= Q^2/(2\psi_0) \end{aligned} \tag{4.10–17}$$

Incidentally, the first approximation to the output of a linear rectifier given by (4.10–16) is interesting in its own right. Fig. 9 shows the low frequency portion of $W_0(f)$ as computed from (4.10–16) when the input noise is uniformly distributed over a narrow frequency band of width β, f_q being the mid-band frequency. h_{11} and h_{02} may be obtained from the curves shown in Fig. 10. In these figures P and x replace Q and y of (4.10–17) in order to keep the notation the same as in Fig. 8 for the square law device. These curves may also be obtained from equations (33) to (43) of Bennett's paper.

The following values are useful for our comparison.

When $x = 0$	When x is large	
$h_{11} = 0$	$h_{11} = 1/\pi$	(4.10–18)
$h_{02} = (2\pi\psi_0)^{-1/2}$	$h_{02} = 1/(\pi Q)$.	

The values for large x are obtained from the asymptotic expansion $(4B - 3)$ given in Appendix $4B$.

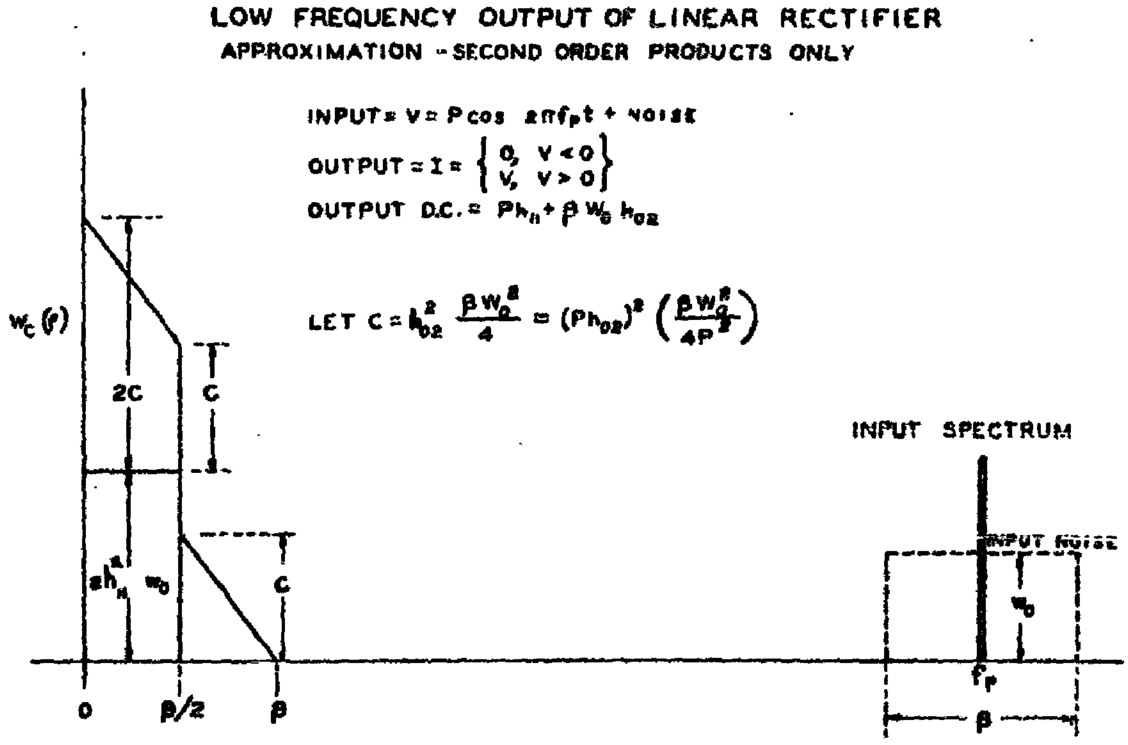

Fig. 9

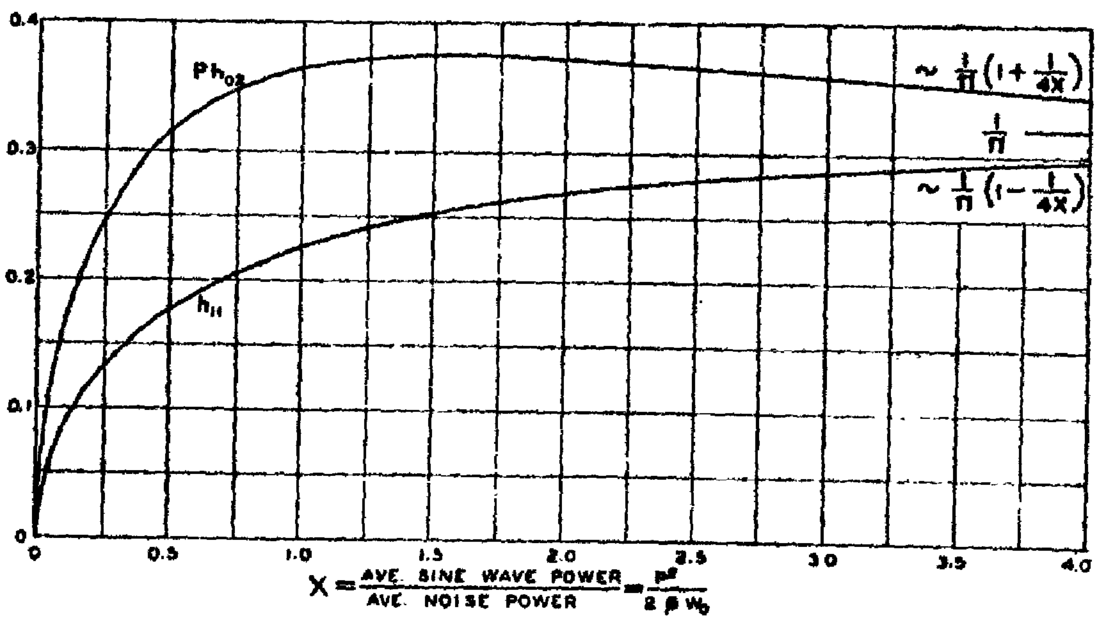

Fig. 10—Coefficients for linear detector output shown on Fig. 9

$$Ph_{02} = \sqrt{\frac{x}{\pi}}\ {}_1F_1(\tfrac{1}{2};\ 1;\ -x) \qquad h_{11} = \frac{1}{2}\sqrt{\frac{x}{\pi}}\ {}_1F_1(\tfrac{1}{2};\ 2;\ -x)$$

We make the first comparison between (4.10–15) and (4.10–16) by letting $Q \to \infty$. It is seen that both reduce to

$$W_c(f) = \frac{1}{\pi^2}\left[w(f_q - f) + w(f_q + f)\right] \qquad (4.10\text{–}19)$$

which shows that the agreement is perfect in this case. Next we let $Q = 0$. The two expressions then give

$$W_c(f) = \frac{1}{A 2\pi\psi_0} \int_{-\infty}^{+\infty} w(x)w(f - x)\, dx$$

where $A = \pi$ for Ragazzini's formula and $A = 4$ for (4.10–16). Thus the agreement is still quite good. The limiting value for (4.10–16) may also be obtained from (4.7–8).

Even if the index of modulation r is not negligibly small it may be shown that when $Q \to \infty$ $W_c(f)$ still approaches the value given by (4.10–19). Ragazzini's formula gives a somewhat larger answer because it includes the additional terms, shown in (4.5–17), which contain $k^2/4$, but this difference does not appear to be serious. If the $Q^2 + 2\psi_0$ in the denominator of (4.10–13) be replaced by $Q^2 + \frac{1}{2}Q^2k^2 + 2\psi_0$ the agreement is improved.

APPENDIX 4A

Table of Non-linear Devices Specified by Integrals

Quite a number of non-linear devices may be specified by integrals of the form

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu}\, du \qquad (4A\text{–}1)$$

where the function $F(iu)$ and the path of integration C are chosen to fit the device.* The table gives examples of such devices. Some important cases cannot be simply represented in this form. An example is the limiter

$$\begin{aligned} I &= -\alpha D, & V &< -D \\ I &= \alpha V, & -D &< V < D \\ I &= \alpha D, & D &< V \end{aligned} \qquad (4A\text{–}2)$$

which may be represented as

$$\begin{aligned} I &= \frac{2\alpha}{\pi} \int_0^\infty \sin Vu \sin Du \frac{du}{u^2} \\ &= -\alpha D + \frac{2\alpha}{2\pi i} \int_C e^{iVu} \sin Du \frac{du}{u^2} \end{aligned} \qquad (4A\text{–}3)$$

where C runs from $-\infty$ to $+\infty$ and is indented downward at the origin. This is not of the form assumed in the theory of Part IV. However it appears that it would not be difficult to extend the theory in the particular case of the limiter.

* Reference 50 cited in Section 4.9.

Non-Linear Devices Specified by Integrals

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu}\, du$$

I	$F(iu)$	C	Type of Device
$I = \alpha V^n$, n integer	$\frac{\alpha\, n!}{(iu)^{n+1}}$	Positive Loop around $u = 0$	nth power device
$I = \alpha(V - B)^n$, n integer	$\frac{\alpha\, n!}{(iu)^{n+1}} e^{-iuB}$	Positive Loop around $u = 0$	nth power device with bias
$I = 0, \quad V < 0$ $I = \alpha V, \quad 0 < V$	$\frac{\alpha}{(iu)^2} = -\frac{\alpha}{u^2}$	Real u axis from $-\infty$ to $+\infty$ with downward indentation at $u = 0$	Linear rectifier cut-off at $V = 0$
$I = 0, \quad V < B$ $I = \alpha(V - B)^\nu$, $V > B$ ν any positive number	$\frac{\alpha\Gamma(\nu + 1)}{(iu)^{\nu+1}} e^{-iuB}$	"	νth power rectifier with bias
$I = 0, \quad V < 0$ $I = \alpha V, \quad 0 < V < D$ $I = \alpha D, \quad D < V$	$\frac{\alpha(1 - e^{-iuD})}{(iu)^2}$	"	Linear rectifier plus limiter
$I = 0, \quad V < 0$ $I = \varphi(V), \quad V > 0$	$F(p) = \int_0^\infty e^{-pt}\varphi(t)\, dt$	"	

APPENDIX 4B

THE FUNCTION ${}_1F_1(a; c; x)$

In problems concerning a sine wave plus noise the hypergeometric function

$$ {}_1F_1(a; c; z) = 1 + \frac{az}{c1!} + \frac{a(a + 1)}{c(c + 1)} \frac{z^2}{2!} + \cdots \qquad \text{(4B–1)}$$

arises. Here we state some of its properties which are of use in the theory of Part IV. Curves of ${}_1F_1(a; c; z)$ are given for $a = -4, -3.5 \cdots, 3.5, 4.0$ and $c = -1.5, -.5, +.5, 1, 1.5, 2, 3, 4$ in the 1938 edition, page 275, of "Tables of Functions", by Jahnke and Emde. A list of properties of the function and other references are also given. In addition to these references we mention E. T. Copson, "Functions of a Complex Variable" (Oxford, 1935), page 260.

If c is not a negative integer or zero

$$ {}_1F_1(a; c; z) = e^z {}_1F_1(c - a; c; -z). \qquad \text{(4B–2)}$$

When $R(z) > 0$ we have the asymptotic expansions

$$
\begin{aligned}
{}_1F_1(a;c;z) &\sim \frac{\Gamma(c)e^z}{\Gamma(a)z^{c-a}}\left[1+\frac{(1-a)(c-a)}{1!z}\right.\\
&\quad\left.+\frac{(1-a)(2-a)(c-a)(c-a+1)}{2!z^2}+\cdots\right]\\
{}_1F_1(a;c;-z) &\sim \frac{\Gamma(c)}{\Gamma(c-a)z^a}\left[1+\frac{a(1+a-c)}{1!z}\right.\\
&\quad\left.+\frac{a(a+1)(1+a-c)(2+a-c)}{2!z^2}+\cdots\right]
\end{aligned}
\qquad (4B\text{–}3)
$$

Many of the hypergeometric functions encountered may be expressed in terms of Bessel functions of the first kind for imaginary argument. The connection may be made by means of the relation[61]

$$
{}_1F_1\left(\nu+\frac{1}{2};2\nu+1;z\right) = 2^{2\nu}\Gamma(\nu+1)z^{-\nu}e^{z/2}I_\nu\left(\frac{z}{2}\right) \qquad (4B\text{–}4)
$$

together with the recurrence relations

	F_{a+}	F_{a-}	F_{c+}	F_{c-}	F
1.	a	$(a-c)$			$c-2a-z$
2.	ac		$(c-a)z$		$-c(a+z)$
3.	a			$1-c$	$c-a-1$
4.		$-c$	$-z$		c
5.		$a-c$		$c-1$	$1-a-z$
6.			$(c-a)z$	$c(c-1)$	$c(1-c-z)$

For example, the first recurrence relation is obtained from line 1 as follows

$$
\begin{aligned}
&aF(a+1;c;z)+(a-c)F(a-1;c;z)\\
&\qquad\qquad+(c-2a-z)F(a;c;z)=0 \qquad (4B\text{–}5)
\end{aligned}
$$

These six relations between the contiguous ${}_1F_1$ functions are analogous to the 15 relations, given by Gauss, between the contiguous ${}_2F_1$ hypergeometric functions and may be derived from these by using

$$
{}_1F_1(a;c;z) = \operatorname*{Limit}_{b\to\infty}\, {}_2F_1\left(a,b;c;\frac{z}{b}\right) \qquad (4B\text{–}6)
$$

A recurrence relation involving two ${}_1F_1$'s of the type (4B–4) may be obtained by replacing a by $a+1$ in the relation given by row four of the table

[61] G. N. Watson, "Theory of Bessel Functions" (Cambridge, 1922), p. 191.

and then eliminating ${}_1F_1(a+1;c;z)$ from this relation and the one obtained from row 3 of the table. There results

$$ {}_1F_1(a;c;z) = {}_1F_1(a;c-1;z) + \frac{za}{c(1-c)} F(a+1;c+1;z) \qquad \text{(4B-7)} $$

Setting ν equal to zero and one in (4B-4) and a equal to $\frac{1}{2}$, c equal to 2 in (4B-7) gives

$$ \begin{aligned} {}_1F_1\left(\frac{1}{2};1;z\right) &= e^{z/2} I_0\left(\frac{z}{2}\right) \\ {}_1F_1\left(\frac{3}{2};3;z\right) &= 4z^{-1}e^{z/2} I_1\left(\frac{z}{2}\right) \\ {}_1F_1\left(\frac{1}{2};2;z\right) &= e^{z/2}\left[I_0\left(\frac{z}{2}\right) - I_1\left(\frac{z}{2}\right)\right] \end{aligned} \qquad \text{(4B-8)} $$

Starting with these relations the relations in the table enable us to find an expression for ${}_1F_1(n+\frac{1}{2};m;z)$ where n and m are integers. A number of these are given in Bennett's paper. In particular, using (4B-2),

$$ {}_1F_1\left(-\frac{1}{2};1;-z\right) = e^{-z/2}\left[(1+z)I_0\left(\frac{z}{2}\right) + zI_1\left(\frac{z}{2}\right)\right]. \qquad \text{(4B-9)} $$

APPENDIX 4C

The Power Spectrum Corresponding to ψ_τ^n

Quite often we encounter the integral

$$ G_n(f) = \int_0^\infty [\psi(\tau)]^n \cos 2\pi f\tau \, d\tau \qquad \text{(4C-1)} $$

where $\psi(\tau)$ is the correlation function corresponding to the power spectrum $w(f)$. From the fundamental relation between $w(f)$ and $\psi(\tau)$ given by (2.1-5),

$$ G_1(f) = \frac{w(f)}{4} \qquad \text{(4C-2)} $$

The expression for the spectrum of the product of two functions enables us to write $G_n(f)$ in terms of $w(f)$. We shall use the following form of this expression: Let $F_r(f)$ be the spectrum of the function $\varphi_r(\tau)$ so that

$$ \varphi_r(\tau) = \int_{-\infty}^{+\infty} F_r(f)e^{2\pi i f\tau}\, df, \quad r = 1, 2 $$

$$ F_r(f) = \int_{-\infty}^{+\infty} \varphi_r(\tau)e^{-2\pi i f\tau}\, dt $$

Then

$$\int_{-\infty}^{+\infty} \varphi_1(\tau)\varphi_2(\tau)e^{-2\pi i f\tau}\,d\tau = \int_{-\infty}^{+\infty} F_1(x)F_2(f-x)\,dx \qquad \text{(4C–3)}$$

i.e., the spectrum of the product $\varphi_1(\tau)\varphi_2(\tau)$ is the integral on the right. If $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are real even functions of τ, (4C–3) may be written as

$$\int_0^{\infty} \varphi_1(\tau)\varphi_2(\tau)\cos 2\pi f\tau\,d\tau = \frac{1}{2}\int_{-\infty}^{+\infty} F_1(x)F_2(f-x)\,dx \qquad \text{(4C–4)}$$

In order to obtain $G_2(f)$ we set $\varphi_1(\tau)$ and $\varphi_2(\tau)$ equal to $\psi(\tau)$. We may then use (4C-4) since $\psi(\tau)$ is an even real function of τ. When $\varphi_r(\tau)$ is an even real function of τ we see, from the Fourier integral for $F_r(f)$, that $F_r(f)$ must be an even real function of f. We therefore set

$$2F_r(f) = w(f), \qquad r = 1, 2$$

and define $w(f)$ for negative f by

$$w(-f) = w(f) \qquad \text{(4C–5)}$$

Equation (4C–4) then gives

$$\begin{aligned} G_2(f) &= \frac{1}{8}\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx \\ &= \frac{1}{8}\int_0^{f} w(x)w(f-x)\,dx \\ &\quad + \frac{1}{4}\int_0^{\infty} w(x)w(f+x)\,dx \end{aligned} \qquad \text{(4C–6)}$$

where in the second equation only positive values of the argument of $w(f)$ appear.

In order to get $G_3(f)$ we set $\varphi_1(\tau)$ equal to $\psi(\tau)$, $2F_1(f)$ equal to $w(f)$, and $\varphi_2(\tau)$ equal to $\psi^2(\tau)$. Then

$$\begin{aligned} F_2(f) &= 2\int_0^{\infty} \varphi_2(\tau)\cos 2\pi f\tau\,d\tau \\ &= 2G_2(f) \end{aligned}$$

and from (4C–4) we obtain

$$\begin{aligned} G_3(f) &= \frac{1}{2}\int_{-\infty}^{+\infty} w(x)G_2(f-x)\,dx \\ &= \frac{1}{16}\int_{-\infty}^{+\infty} w(x)\,dx \int_{-\infty}^{+\infty} w(y)w(f-y)\,dy \end{aligned} \qquad \text{(4C–7)}$$

Equation (4C–7) suggests that we may write the expression for $G_2(f)$ as

$$G_2(f) = \frac{1}{2}\int_{-\infty}^{+\infty} w(x)G_1(f - x)\,dx \tag{4C-8}$$

This is seen to be true from (4C–2) and (4C–6). In fact it appears that

$$G_n(f) = \frac{1}{2}\int_{-\infty}^{+\infty} w(f - x)G_{n-1}(x)\,dx \tag{4C-9}$$

might be used for a step by step computation of $G_n(f)$.

We now consider $G_n(f)$ for the case of relatively narrow band pass filters. As examples we take filters whose characteristics give the following $w(f)$'s and $\psi(\tau)$'s

TABLE 1

Filter	$w(f)$ for $f > 0$	$\psi(\tau)$
a	$\frac{\psi_0}{\sigma\sqrt{2\pi}} e^{-(f-f_0)^2/2\sigma^2}$	$\psi_0 e^{-2(\pi\sigma\tau)^2} \cos 2\pi f_0 \tau$
b	$\frac{\psi_0 \alpha}{\pi} \frac{1}{\alpha^2 + (f - f_0)^2}$	$\psi_0 e^{-2\pi\alpha\|\tau\|} \cos 2\pi f_0 \tau$
c	$w(f) = w_0 = \psi_0/\beta$ for $f_0 - \frac{\beta}{2} < f < f_0 + \frac{\beta}{2}$; $w(f) = 0$ elsewhere	$\psi_0 \frac{\sin \pi\beta\tau}{\pi\beta\tau} \cos 2\pi f_0 \tau$

We shall refer to these filters as Filter a, Filter b, and Filter c, respectively. All have f_0 as the mid-frequency of the pass band. The constants have been chosen so that they all pass the same average power when a wide band voltage is applied:

$$\psi_0 = \int_0^\infty w(f)\,df = \text{mean square value of } I(t) \text{ or } V(t)$$

and it is assumed that $f_0 \gg \sigma$, $f_0 \gg \alpha$, $f_0 \gg \beta$ so that the pass bands are relatively narrow.

Expressions for $G_n(f)$ corresponding to several values of n are given in Table 2. When $n = 1$, $G_1(f)$ is simply $w(f)/4$. $G_2(f)$ is obtained by setting $n = 2$ in the definition (4C–1) for $G_n(f)$, squaring the $\psi(\tau)$'s of Table 1, and using

$$\cos^2 2\pi f_0\tau = \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f_0\tau$$

Table 2

$G_n(f)$	Filter a	Filter b
$G_1(f)$	$\frac{\psi_0}{4\sigma\sqrt{2\pi}} e^{-(f-f_0)^2/2\sigma^2}$	$\frac{\alpha\psi_0}{4\pi} \frac{1}{\alpha^2 + (f-f_0)^2}$
$G_2(f)$	$\frac{\psi_0^2}{8\sigma\sqrt{4\pi}} [2e^{-f^2/4\sigma^2} + e^{-(f-2f_0)^2/4\sigma^2}]$	$\frac{2\alpha\psi_0^2}{8\pi}\left[\frac{2}{4\alpha^2+f^2} + \frac{1}{4\alpha^2+(f-2f_0)^2}\right]$
$G_3(f)$	$\frac{\psi_0^3}{16\sigma\sqrt{6\pi}} [3e^{-(f-f_0)^2/6\sigma^2} + e^{-(f-3f_0)^2/6\sigma^2}]$	$\frac{3\alpha\psi_0^3}{16\pi}\left[\frac{3}{9\alpha^2+(f-f_0)^2} + \frac{1}{9\alpha^2+(f-3f_0)^2}\right]$
$G_4(f)$	$\frac{\psi_0^4}{32\sigma\sqrt{8\pi}} [6e^{-f^2/8\sigma^2} + 4e^{-(f-2f_0)^2/8\sigma^2} + e^{-(f-4f_0)^2/8\sigma^2}]$	$\frac{4\alpha\psi_0^4}{32\pi}\left[\frac{6}{16\alpha^2+f^2} + \frac{4}{16\alpha^2+(f-2f_0)^2} + \frac{1}{16\alpha^2+(f-4f_0)^2}\right]$
$G_n(f)$ n odd f small	0	0
$G_n(f)$ n even f small	$\frac{\psi_0^n n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \frac{e^{-f^2/2n\sigma^2}}{2^{n+1}\sigma\sqrt{2n\pi}}$	$\frac{\psi_0^n n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)! 2^n} \frac{1}{2^{n+1}\pi n\alpha} \frac{1}{1+\left(\frac{f}{n\alpha}\right)^2}$
$G_n(f)$ n even n large f small	$\frac{1}{2\pi\sigma n} e^{-f^2/2n\sigma^2}$	$\frac{2}{\alpha(2\pi n)^{3/2}} \frac{1}{1+\left(\frac{f}{n\alpha}\right)^2}$
Filter c $G_1(f)$	$\frac{\psi_0}{4\beta}$ when $f_0 - \frac{\beta}{2} < f < f_0 + \frac{\beta}{2}$ 0 elsewhere	Filter c $G_2(f)$ $\frac{\psi_0^2}{4\beta}\left(1-\frac{f}{\beta}\right)$ when $0 \le f \le \beta$ $\frac{\psi_0^2}{8\beta^2}(f - 2f_0 + \beta)$ " $2f_0 - \beta \le f \le 2f_0$ $\frac{\psi_0^2}{8\beta^2}(2f_0 + \beta - f)$ " $2f_0 \le f \le 2f_0 + \beta$

The expression for $G_2(f)$ given in Table 2 corresponding to Filter c is exact. The expressions for Filters a and b give good approximations around $f = 0$ and $f = 2f_0$ where $G_2(f)$ is large. However, they are not exact because terms involving $f + 2f_0$ have been omitted. It is seen that all three G_2's behave in the same manner. Each has a peak symmetrical about $2f_0$ whose width is twice that of the original $w(f)$, is almost zero between 0 and $2f_0$, and rises to a peak at 0 whose height is twice that at $2f_0$.

$G_3(f)$ is obtained by cubing the $\psi(\tau)$ given in Table 1 and using

$$\cos^3 2\pi f_0\tau = \tfrac{3}{4} \cos 2\pi f_0\tau + \tfrac{1}{4} \cos 6\pi f_0\tau.$$

From the way in which the cosine terms combine with $\cos 2\pi f\tau$ in (4C-1) we see that $G_3(f)$, for our relatively narrow band pass filters, has peaks at f_0 and $3f_0$, the first peak being three times as high as the second. The expressions given for $G_3(f)$ and $G_4(f)$ are approximate in the same sense as are those for $G_2(f)$. It will be observed that the coefficients within the brackets, for Filters a and b, are the binomial coefficients for the value of n concerned. Thus for $n = 2$, they are 2 and 1, for $n = 3$ they are 3 and 1, and for $n = 4$ they are 6, 4, and 1.

The higher $G_n(f)$'s for Filters a and b may be computed in the same way. The integrals to be used are

$$\int_0^\infty e^{-2n(\pi\sigma\tau)^2} \cos 2\pi f\tau \, d\tau = \frac{e^{-f^2/2n\sigma^2}}{2\sigma\sqrt{2n\pi}}$$

$$\int_0^\infty e^{-2n\pi\alpha\tau} \cos 2\pi f\tau \, d\tau = \frac{1}{2\pi} \frac{n\alpha}{n^2\alpha^2 + f^2}$$

In many of our examples we are interested only in the values $G_n(f)$ for f near zero, i.e., only in that peak which is at zero. It is seen that $G_n(f)$ has such a peak only when n is even, this peak arising from the constant term in the expansion

$$\cos^{2k} x = \frac{1}{2^{2k-1}} \left[\cos 2kx + 2k \cos 2(k-1)x + \frac{(2k)(2k-1)}{2!} \cos 2(k-2)x \right.$$
$$\left. + \cdots + \frac{(2k)!}{(k-1)!(k+1)!} \cos 2x + \frac{(2k)!}{k!k!2} \right]$$

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